Title
Geometric Control Theoretic Formulation Applied to the Analysis of Pitching and Plunging Airfoils

Author
Pla Olea, Laura

Publication Date
2019

UNIVERSITY OF CALIFORNIA,
IRVINE

Geometric Control Theoretic Formulation Applied to the Analysis of Pitching and Plunging Airfoils

THESIS

submitted in partial satisfaction of the requirements
for the degree of

MASTER OF SCIENCE

in Mechanical and Aerospace Engineering

by

Laura Pla Olea

Thesis Committee:
Professor Haithem E. Taha, Chair
Professor Roger H. Rangel
Professor James Bobrow

2019

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ACKNOWLEDGMENTS

First of all, I would like to thank the Department of Mechanical and Aerospace Engineering for accepting me and letting me be part of their community here. In this last year, I have solidified and acquired new knowledge thanks to the professors and the community here at University of California, Irvine.

I would also like to thank Pete Balsells, the Balsells Fellowship program and its director, Professor Roger Rangel for giving me the opportunity to come to the University of California, Irvine and making this dream come true. It would not have been possible without the support of The Balsells Fellowship Foundation, the *Generalitat de Catalunya* and The Henry Samueli School of Engineering. They have helped not only me but many others to grow professionally and personally.

My advisor, Haithem E. Taha, thank you for helping me in this project. You have been my mentor for the past year and my success can be greatly atributted to your patience and motivation. You have introduced me to unsteady aerodynamics and geometric control, two fields that were unknown to me but have definitely captured my interest.

I must thank my parents, for their unconditional support. Thank you for believing in me for as long as I can remember, even when I did not.

I also want to acknowledge the help of my laboratory mates. The long work hours were easier and more enjoyable thanks to your company.

Last but not least, I should thank my friends: the ones that have always been there and gave me support from home through long calls; and the ones that I met here in California, some of whom are also Balsells fellows, that shared this experience with me and made me feel like home at 10,000 km from it.

ABSTRACT OF THE THESIS

Geometric Control Theoretic Formulation Applied to the Analysis of Pitching and Plunging Airfoils

By

Laura Pla Olea

Master of science in Mechanical and Aerospace Engineering

University of California, Irvine, 2019

Professor Haithem E. Taha, Chair

The unsteady aerodynamics of a two-dimensional pitching-plunging airfoil are modeled using a state space system. This computationally inexpensive model, simpler than the current classical theories of unsteady aerodynamics, is an output-based model, providing only the dynamics of certain output variables, without reconstructing the entire flow field.

Due to the simplicity of the system, it allows for the analytical study of the behavior of four outputs: lift, drag, circulation and point of separation. In order to capture the nonlinearities of the complex unsteady flow, the four outputs are analyzed using some of the tools provided by the geometric control theory. The results are then studied in a fluid dynamics framework to determine if the unsteady motion of the airfoil can provide an enhancement of the mentioned parameters.

Chapter 1

Introduction

1.1 Motivation

More than a hundred years have passed since the first flight by the Wright brothers in Kitty Hawk, North Carolina. Since that December of 1903 airplanes have improved in several ways, thanks to the advances in aerodynamics, structures and materials. However, looking at the models of the last decades, the design appears to be saturated, all airplanes having a similar shape.

Airplanes have fixed wings, and are perfectly designed for steady conditions and small angles of attack. Nonetheless, it seems that it is not possible to further improve this design by relying on steady linear aerodynamics as it has been done for the past hundred years. This field has been thoroughly studied and exploited, so it seems difficult that it can generate innovative improvements that can lead to a revolutionary design.

Therefore, a new approach is required to change the basic shape of airplanes. Looking at nature, it looks like the solution may be flapping propulsion. As opposed to the approach taken by humans, insects and birds rely on unsteady aerodynamics and high angles of attack to truly exploit their aerodynamic capabilities. Due to its complexity, this type of flight has not been as studied as linear aerodynamics. Luckily, geometric control theory can provide some tools to analyze the characteristics of this nonlinear flow.

1.2 Previous work

1.2.1 Unsteady aerodynamics

The earliest studies of unsteady aerodynamics appeared in the 1920s, but it wasn't until 1935 that the harmonic case was addressed by Theodorsen [1]. In his famous paper, Theodorsen studied the lift and aerodynamic moment of a plunging and oscillating airfoil with a flap. Restricting his study to potential flow and harmonic motion, and applying the Kutta condition, he solved the problem by defining a function based on the Bessel functions of first and second kind.

Another outstanding effort was that done by Wagner [2] ten years before. He solved in the context of potential flow the time evolution of the lift in an airfoil that was suddenly started from rest. Joined by Theodorsen, they hold the main classical theories in unsteady aerodynamics, the former in the time domain and the latter in the frequency domain.

Aside from this analytical achievements, several experiments were also carried out. In 1957 Rainey [3] measured the force and moment of a pitching airfoil as a function of the reduced frequency. He found that pitching oscillations modify the slope of the lift force and pitching moment of the airfoil, but their magnitude and phase depend on the mean angle of attack, Mach number and reduced frequency.

Skipping forward many decades, the effect of unsteady aerodynamics has also been experimentally found in the direction of the free-stream velocity. An interesting result was published in 2004, when Vandenberghe et al. [4] found that a pure plunging oscillatory airfoil at zero forward speed spontaneously generated a net thrust force if it exceeded a critical frequency. This generation of a net force as a result of a zero mean oscillation is known as symmetry breaking, and it appears in several unsteady aerodynamics cases.

The effect of pitching and plunging on an airfoil has been studied in several cases, mostly with an experimental approach. For example, Rival and Tropea [5] analyzed the wake of pure-pitching and pure-plunging profiles as well as of their combined motion through smoke visualizations and force measurements. They found the geometry of the wake as well as the value of the measured aerodynamic forces to deeply depend on the reduced frequency of the motion and the angle of attack.

Along the same lines, Rival et al. [6] experimentally studied the mechanism of vortex detachment in plunging profiles with different leading edge geometries at zero angle of attack. On a different side, Heathcote and Gursul [7] analyzed the deflection of the vortex pairs that form at the trailing edge of a plunging airfoil in a water tank.

As it can be seen, the fluid dynamics community has discovered through experimental observation the unconventional dynamical behaviors that appear when the nonlinearities of the flow are exploited. However, no substantial effort has been done in order to develop an analytical model simpler than those of Theodorsen and Wagner that can predict these outcomes. As it has been stated before, it can be attributed to the lack of tools to analyze higher order effects in nonlinear systems, but thanks to geometric control theory this complication can be overcome.

1.2.2 Geometric control

The geometric control theory is an application of the mathematically complex differential geometry to control theory [8]. Its purpose is to study nonlinear dynamical systems that evolve on curvy spaces named manifolds.

Geometric control theory was first developed by Brockett [9, 10, 11, 12] and Hector Sussmann [13, 14, 15] in the 1970s. Since then, it has been used in the study of the dynamics of several nonlinear systems in different engineering applications, such as robotics [16] or spacecraft attitude control [17]. However, to the author's knowledge, this theory has not been employed in the resolution of fluid mechanics problems.

Its approach is particularly useful in the resolution of nonlinear systems due to its capacity to capture higher order effects. This characteristic allows for the discovery of unconventional force and stabilization mechanisms [8], like the ones found in the experimental observations mentioned in the previous section.

1.3 Thesis overview

The following project presents a model to characterize the unsteady flow around a two-dimensional airfoil in the form of a state space model. In order to do that, the airfoil is modeled as a flat plate, and the motion is simplified to pitching and plunging movements. This reduced order model (ROM) does not require the reconstruction of the entire flow field, it only provides the dynamics of a few output variables of interest: lift, drag, circulation and point of separation.

The simplicity of this model allows the obtainment of analytical expressions that can be further analyzed to predict the behavior of the airfoil. Thus, once the mentioned variables are derived, they are analyzed with some of the tools provided by geometric control theory. The objective of this thesis is to determine if it is possible to enhance the performance of the airfoil (e.g. increase the lift or generate thrust) due to nonlinear interactions between the pitching and plunging motions in an unconventional way. In other words, the main purpose of this work is to predict the unsteady phenomena described in the previous paragraphs and others that may not have been found yet.

The thesis is divided in several chapters. In Chapter 1, the topic of study is introduced, and a brief review of the previous work done in the areas of unsteady aerodynamics and geometric control theory is conducted. Chapter 2 explains the foundations of unsteady aerodynamics through the classical theories of Theodorsen and Wagner. In addition, it introduces a state space model that simplifies the calculations of the previously mentioned theories and extends their use to more complicated cases. This model studies the lift of a two-dimensional airfoil, and is the basis of the system that is going to be analyzed in the subsequent chapters. A very brief presentation of the geometric control concepts that are employed in the thesis can be found in Chapter 3. The core of this work is in Chapter 4, which proposes a state space model that characterizes the lift of a pitching and plunging airfoil. This system is further expanded in this same chapter to account for the drag, circulation and point of separation. Later on, all these outputs are studied using the geometric control tools defined in Chapter 3, with the purpose of predicting some non-intuitive nonlinear outcomes that may improve the overall performance of the airfoil. Finally, Chapter 5 serves as a summary of the results obtained in this thesis.

Chapter 2

Aerodynamics

2.1 Theodorsen

Theodore Theodorsen is one of the pioneers of the classical unsteady aerodynamics theory and his work is used as a reference in most of the studies in this field. Theodorsen [1] studied the aerodynamic forces on an oscillating airfoil with three independent degrees of freedom: pitching, plunging and aileron flapping. The main assumptions of his theory are potential flow and the use of the no-penetration boundary condition, the Kutta condition and Kelvin's theorem of conservation of circulation. He also considered the wake of the airfoil to be horizontal, in the direction of the free-stream velocity.

Theodorsen's objective was to compute the lift of the airfoil when it was subject to this unsteady motion. He found that it could be divided in two different contributions:

1. Non-circulatory lift: It does not generate net circulation. It only depends on the instantaneous acceleration of the profile.

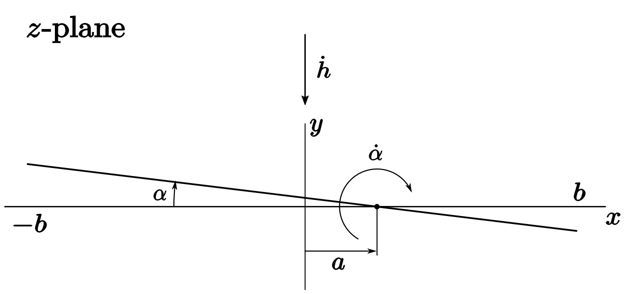
2. Circulatory lift: It is due to the surface of discontinuity behind the wing, the wake. It is similar to the effect that appears in steady aerodynamics.

2.1.1 Non-circulatory lift

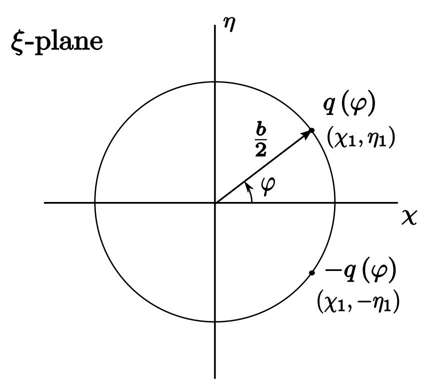
In the classical aerodynamics theory, the airfoil is always replaced with vortices that generate the vorticity (circulation) needed to have lift. Nonetheless, since this case requires a singularity without circulation, the usual vortices are replaced by sources and sinks.

Due to the complexity of the airfoil geometry, the wing is approximated to a flat plate with the same chord, $c = 2b$, as it can be seen in figure 2.1a. See that even though Theodorsen derived his theory for a pitching-plunging-flapping airfoil in this case only the pitching and plunging motions are considered, since the flapping motion is out of the scope of this thesis.

After the definition of the problem, since the most studied shape in potential flow is a cylinder, the airfoil is replaced by a cylinder of radius b using the Joukowsky transformation. In this transformation, the original plate is situated in the complex z-plane, where $z = x+iy$, whereas the cylinder in which the calculations are performed is in the ξ-plane, where $\xi = \chi + i\eta$.



(a) Parameters of the airfoil in the z-plane.



(b) Conformal representation of the wing profile by a circle.

Figure 2.1: Transformation of a flat plat in the z-plane into a circle in the ξ-plane.

Once the cylinder is defined, a source of strength q is placed at the point on its surface (χ_1, η_1) and a sink of strength $-q$ is placed at $(\chi_1, -\eta_1)$, as indicated in figure 2.1b. The strength of both sink and source depend on their position.

However, since the airfoil is a continuous surface, the upper half of the circle has to be defined by a continuous distribution of sources on it, and the lower surface by a continuous distribution of sinks. Therefore, $q(\varphi)$ being the strength distribution of these sources and sinks as a function of their angular position φ, the total complex potential of the circle is

$$F_{NC} = \frac{q(\varphi)\frac{b}{2}d\varphi}{2\pi}\left[\ln\left(\xi - \frac{b}{2}\mathrm{e}^{i\varphi}\right) - \ln\left(\xi - \frac{b}{2}\mathrm{e}^{-i\varphi}\right)\right]$$

and the complex velocity

$$\bar{w}_{NC} = \frac{dF_{NC}}{d\xi} = \frac{q(\varphi)\frac{b}{2}d\varphi}{2\pi}\left[\frac{1}{\xi - \frac{b}{2}\mathrm{e}^{i\varphi}} - \frac{1}{\xi - \frac{b}{2}\mathrm{e}^{-i\varphi}}\right]$$

Applying the change of coordinates $\xi = r\mathrm{e}^{i\theta}$ and after some algebraic manipulations

$$\bar{w}_{NC} = \frac{q(\varphi)\frac{b^2}{4}d\varphi}{2\pi}\mathrm{e}^{-i\theta}\frac{2i\sin\varphi}{r^2\mathrm{e}^{i\theta} - \frac{b}{2}r2\cos\varphi + \frac{b^2}{4}\mathrm{e}^{-i\theta}} = \mathrm{e}^{-i\theta}\left(v_r - iv_\theta\right)$$

Now, defining the real and imaginary parts of the denominator as $R = r^2 + \frac{b^2}{4}\cos\theta - br\cos\varphi$ and $I = \left(r^2 - \frac{b^2}{4}\right)\sin\theta$ respectively, the normal and tangential velocities of the flow can be determined. The velocities at a point situated at a distance r from the origin and at an angle θ are

$$v_{\theta_{NC}}(r,\theta) = -\frac{q(\varphi)\frac{b^2}{4}\sin\varphi R(\theta,\varphi)d\varphi}{\pi\left(R^2 + I^2\right)}$$

$$v_{r_{NC}}(r,\theta) = \frac{q(\varphi)\frac{b^2}{4}\sin\varphi I(\theta,\varphi)d\varphi}{\pi\left(R^2 + I^2\right)}$$

However, these are the velocities due to a point source and a point sink located on the surface of the cylinder at an angle φ. If there is a whole distribution of them around the cylinder surface they have to be integrated from 0 to π. In doing so, the velocities that can be found in the cylinder due to the non-circulatory contributions of the flow (sources and sinks distribution) are

$$v_{\theta_{NC}}(\theta) = -\frac{1}{2\pi}\int_0^{\pi} \frac{q(\varphi)\sin\varphi d\varphi}{\cos\theta - \cos\varphi} \tag{2.1}$$

$$v_{r_{NC}}(\theta) = \frac{q(\theta)}{2} \tag{2.2}$$

Unfortunately, the source strength distribution is still undefined, so, in order to find it, the no-penetration boundary condition is applied to the surface of the cylinder

$$v_{\perp} = v_{motion}$$

In other words, the normal velocity of the airfoil/cylinder wall has to be equal to the normal velocity of the flow that surrounds it, so that there is no flow that goes through the surface of study. This result is obtained by equating the normal velocity at the cylinder surface to the normal velocity of motion. The movement of the airfoil is given by the expression

$$v_{motion}(x) = -U\sin\alpha - \dot{h}\cos\alpha - \dot{\alpha}\,(x - a)$$

where U is the free-stream velocity, α the angle of attack, $\dot{h}$ the plunging velocity, and a the position of the pitching axis.

Equating it with the normal velocity at the surface (equation 2.2), the source distribution can be found

$$q(\theta) = 2v_{r_{NC}} = 4v_{motion}(\theta)\sin\theta = 4\left[-U\sin\alpha - \dot{h}\cos\alpha - \dot{\alpha}\,(b\cos\theta - a)\right]\sin\theta$$

And from this expression the tangential velocity is obtained with equation 2.1

$$v_{\theta_{NC}}(\theta) = 2\left[\left(U\sin\alpha + \dot{h}\cos\alpha - \dot{\alpha}a\right)\cos\theta + \frac{\dot{\alpha}b}{4}\cos\left(2\theta\right)\right]$$

Once the tangential velocity is known the potential of the cylinder can be found

$$\phi_{NC}(\theta) = -\frac{b}{2}\int_0^{\pi} v_{\theta_{NC}}(\varphi)d\varphi = b\left[\left(U\sin\alpha + \dot{h}\cos\alpha - \dot{\alpha}a\right)\sin\theta + \frac{\dot{\alpha}b}{4}\sin\left(2\theta\right)\right]$$

Therefore, the only thing left to compute is the lift generated due to the non-circulatory flow. In order to find the pressure as a function of the potential, the unsteady Bernoulli equation is used

$$p - p_{\infty} = -\rho\left[U_{\infty}\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t}\right]$$

where p is the pressure and ρ the density of the flow.

The lift can be obtained as an integration of the pressure field that surrounds the cylinder

$$l = 2\rho\left[\int_{-\frac{b}{2}}^{\frac{b}{2}}\frac{\partial\phi}{\partial t}dx + U_{\infty}\left(\phi_{TE} - \phi_{LE}\right)\right]$$

where ϕ_{TE} and ϕ_{LE} are the velocity potentials at the trailing edge and the leading edge respectively. Assuming $\phi_{TE}(\theta) = 0$, and inserting the potential of the cylinder on the lift equation, the non-circulatory lift is obtained

$$l_{NC} = \pi\rho b^2\left(U\dot{\alpha} + \ddot{h} - \ddot{\alpha}a\right) = -m_{added}a_{\perp_{1/2}} \tag{2.3}$$

As it can be seen, this non-circulatory lift is proportional to the instantaneous acceleration of the midpoint of the airfoil multiplied by the added mass $m_{added} = \pi\rho b^2$. This force can be interpreted as an extra acceleration that has to be added to move the airfoil in the air. In other words, the force needed to move the airfoil in vacuum is equal to its mass times the acceleration; however, the force needed to move it on the air is bigger, because in order to move the airfoil it is necessary to accelerate the air surrounding it. The added mass term accounts for the mass of an air cylinder whose radius is the half chord. It can be seen as an inertial force.

Another important remark about the non-circulatory lift is that it is given by an algebraic expression. That is, *it does not have any lag*, a change in the motion of the airfoil will lead to an instantaneous variation of the non-circulatory lift.

Finally, it should be pointed out that the Kutta condition is not imposed in the derivation of the expression of the non-circulatory lift.

2.1.2 Circulatory lift

In order to account for the circulation in the airfoil, a vortex of strength Γ is added on the wake of the cylinder. Nonetheless, in order to satisfy Kelvin's Theorem of the conservation of the circulation, another vortex of the same strength is added, but bounded to the cylinder as indicated in figure 2.2. If the wake vortex is located at a point X, to maintain the cylinder surface as a streamline, the location of bounded vortex has to be

$$d = \frac{b^2}{4X}$$

Once all the singularities are defined, their strength can be found by ensuring that the flow satisfies the Kutta condition. To do so, the sum of the tangential velocity due to the non-circulatory contribution and the tangential velocity due to the circulatory contribution at the trailing edge has to be equal to zero.

Therefore, it is necessary to compute the tangential velocity due to the circulatory contribution. The complex potential of a cylinder with a vortex at $(X, 0)$ and another one at $(d, 0)$

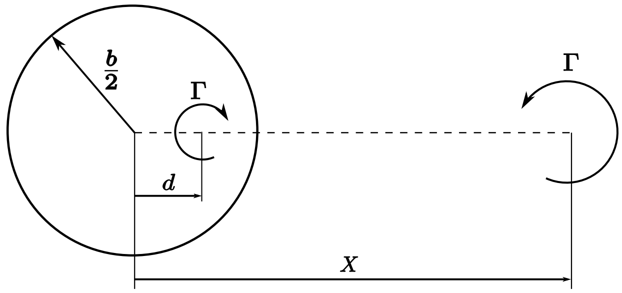


Figure 2.2: Location of the bounded and wake vortex.

is

$$F_C = \frac{\Gamma}{2\pi i}\left[\ln\left(\xi - X\right) - \ln\left(\xi - d\right)\right]$$

And, after the proper calculations, similar to those performed in the previous section, the tangential velocity is obtained

$$v_{\theta_C}(\theta) = -\frac{\Gamma}{b\pi}\frac{X^2 - \frac{b^2}{4}}{X^2 + \frac{b^2}{4} - bX\cos\theta} \tag{2.4}$$

which is used in the application of the Kutta condition at the trailing edge

$$v_{\theta_{NC}}\left(\theta = 0\right) + v_{\theta_C}\left(\theta = 0\right) = 0$$

After the substitution of the velocities by their expressions and applying the Joukowsky transformation $z = \xi + \frac{b^2}{\xi}$, the Kutta condition leads to

$$-2v_{3/4} - \frac{1}{\pi b}\int_b^{\infty}\gamma_w(z,t)\sqrt{\frac{z+b}{z-b}}dz = 0$$

where $v_{3/4}$ is the velocity at the tree-quarter chord point, and γ_w is the unknown vortex distribution of the wake, that has to be integrated from b to ∞ to account for the effects of the whole wake. That is, instead of being approximated by just one vortex, the wake is formed by a continuous distribution of vortices that goes from the trailing edge of the airfoil to infinity. As it was previously mentioned, this wake is assumed to be one-dimensional, in the direction of the free-stream velocity.

Rearranging the equation with the small-angle approximation ($\alpha_{3/4} = -\frac{v_{3/4}}{U}$) the Wagner equation [2] is obtained

$$\frac{1}{Ub}\int_b^\infty \gamma_w(z,t)\sqrt{\frac{z+b}{z-b}}dz = \underbrace{2\pi\alpha_{3/4}(t)}_{C_{L_{QS}}(t)} \tag{2.5}$$

where $\alpha_{3/4}$ is the angle of attack at the three quarter chord point, and $C_{L_{QS}}$ is the quasi-steady lift coefficient of the airfoil.

Going back to the circulatory tangential velocity, the lift can be obtained with the same approach used in the computation of the non-circulatory component. So, from equation 2.4 the potential can be obtained

$$\phi_c(\theta) = \frac{\Gamma}{\pi}\left[\frac{\pi}{2} - \arctan\left(\sqrt{\frac{z+b}{z-b}}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)\right]$$

and applying the unsteady Bernoulli equation, the lift reads

$$l_C(t) = \rho\int_b^\infty \frac{\gamma_w(z,t)z}{\sqrt{z^2-b^2}}dz$$

However, since the value of the quasi-steady circulation is known from the Wagner expression 2.5, the circulatory lift can be written as $l_C(t) = l_C(t)\Gamma_{QS}/\Gamma_{QS}$, leading to

$$l_C(t) = \rho U\Gamma_{QS}(t)\frac{\int_b^\infty \frac{\gamma_w(z,t)z}{\sqrt{z^2-b^2}}dz}{\int_b^\infty \gamma_w(z,t)\sqrt{\frac{z+b}{z-b}}dz} = l_{QS}(t)\frac{\int_b^\infty \frac{\gamma_w(z,t)z}{\sqrt{z^2-b^2}}dz}{\int_b^\infty \gamma_w(z,t)\sqrt{\frac{z+b}{z-b}}dz} \tag{2.6}$$

where l_{QS} is the quasi-steady lift.

This expression does not have an analytical solution due to the unknown wake vorticity distribution γ_w. In order to simplify the results to reach an analytical expression, Theodorsen assumed an harmonic movement of the airfoil

$$\alpha_{3/4} = \bar{\alpha}\mathrm{e}^{i\omega t}$$

Assuming linearity between the airfoil motion and the distribution of the vorticity in its wake

$$\gamma_w(z,t) = \overline{\gamma_w(z)}\mathrm{e}^{i\omega t}$$

Introducing this expression into equation 2.6, after some algebraic modifications, the circulatory lift as it was defined by Theodorsen is obtained

$$l_C(t) = l_{QS}(t)C\left(k\right) \tag{2.7}$$

where $C\left(k\right)$ is the Theodorsen function, a complex function that depends only on the reduced frequency of the motion $k = \frac{\omega b}{U}$. It is defined in the following way

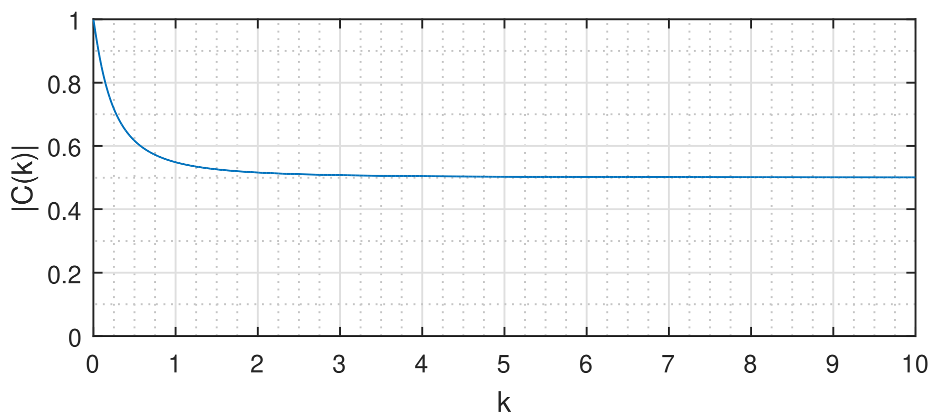
$$C\left(k\right) = \frac{H_1^{(2)}\left(k\right)}{H_1^{(2)}\left(k\right) + iH_0^{(2)}\left(k\right)} = \frac{\int_1^{\infty} \mathrm{e}^{-ik\eta}\frac{\eta}{\sqrt{\eta^2-1}}d\eta}{\int_1^{\infty} \mathrm{e}^{-ik\eta}\sqrt{\frac{\eta+1}{\eta-1}}d\eta}$$

where $H_n^{(m)}$ is the Hankel function of m^{th} kind of order n.

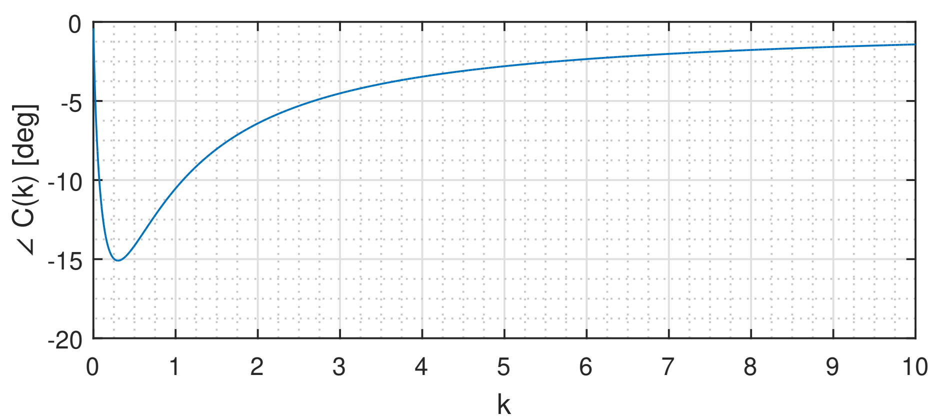
This complex function acts on the motion of the wing. For example, if the airfoil undergoes a pure-plunging motion given by $h = \bar{h}\sin\left(kt\right) = \mathrm{Im}\left[\bar{h}e^{jkt}\right]$, its multiplication by the Theodorsen function results in:

$$C\left(k\right)h(t) = \bar{h}\sqrt{F^2+G^2}\sin\left(kt + \arctan\frac{G}{F}\right)$$

The Theodorsen function is only valid for sinusoidal motion with small amplitudes. It attenuates the lift force as a function of the reduced frequency k. As it can be seen in figure 2.3, $C(k)$ is real and equal to unity for the steady case ($k = 0$), but as k increases, the magnitude of the function becomes $|C(k)| = 1/2$. In other words, for a sudden change in the angle of attack the lift instantaneously acquires half of its steady value. This behavior is illogical in a dynamical system, and represents a limitation in Theodorsen's theory.



(a) Magnitude of Theodorsen function.



(b) Phase of Theodorsen function.

Figure 2.3: Theodorsen function.

2.2 Wagner

Years before Theodorsen [1] proposed his model to obtain the aerodynamic loads of an airfoil as a function of the frequency of its motion, Wagner [2] solved the lift response of a flat plate due to a step input, or in other words, he solved the indicial response problem.

It is easier to understand the relation between the Theodorsen and Wagner functions by reaching Wagner's result from Theodorsen's. Therefore, the derivation of the function proposed in this section is different from that of Wagner's original work [2].

Looking at the solution given by Theodorsen (equation 2.15) a question comes to mind: what occurs when the motion of the wing is not harmonic? The resolution of unsteady problems cannot always be reduced to simple harmonic movements. Thus, if the airfoil follows any other type of motion it may be better to approach the problem from the time domain perspective instead of the frequency one proposed by Theodorsen.

Therefore, it may be necessary to transform the expression of the circulatory lift from the frequency domain into the time domain. However, it is first necessary to express all the variables implied in the calculation of the lift in the frequency domain. Since the time-distribution of the angle of attack at the three-quarter chord point is known, it is directly transformed into the frequency domain with the Fourier transform

$$\alpha_{3/4}(\omega) = \int_{-\infty}^{\infty} \alpha_{3/4}(t)\mathrm{e}^{-i\omega t}dt$$

Then, it can be introduced into the circulatory lift expression of Theodorsen

$$l_C(\omega) = \rho U^2 b 2\pi \alpha_{3/4}(\omega) C\left(k\right)$$

which can be transformed back into the time domain

$$l_C(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} l_C(\omega)\mathrm{e}^{i\omega t}d\omega \tag{2.8}$$

As it was previously stated at the beginning of this section, Wagner [2] computed the lift response as a reaction to a step input. For example, the result of changing the angle of attack instantly from zero to a certain value α

$$\alpha_{3/4}(t) = \begin{cases} \alpha & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In this case the Fourier transform of the angle of attack becomes

$$\alpha_{3/4}(\omega) = \frac{\alpha}{i\omega}$$

Inserting this angle of attack distribution into equation 2.8, the circulatory lift in the time domain is obtained

$$l_C(t) = \rho U^2 b 2\pi\alpha\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{C\left(k\right)}{k}dk = l_{QS}\int_{-\infty}^{\infty}\frac{C\left(k\right)}{k}dk = l_{QS}W\left(s\right) \tag{2.9}$$

where $W\left(s\right)$ is the Wagner function, and it only depends on the value of the non-dimensional time $s = \frac{tU}{b}$. It is equivalent to the Theodorsen function but in the time domain; in fact, as it has been proved in this section, they are a Fourier pair [18]. In fact, their only difference is that Theodorsen solves the lift for harmonic motion whereas in Wagner's case the input is a step function.

This function accounts for the lag that appears between the change in the angle of attack until the lift builds up. In other words, if the angle of attack is changed abruptly the lift does not reach its static value instantly, but it takes some time to evolve to it. However, as

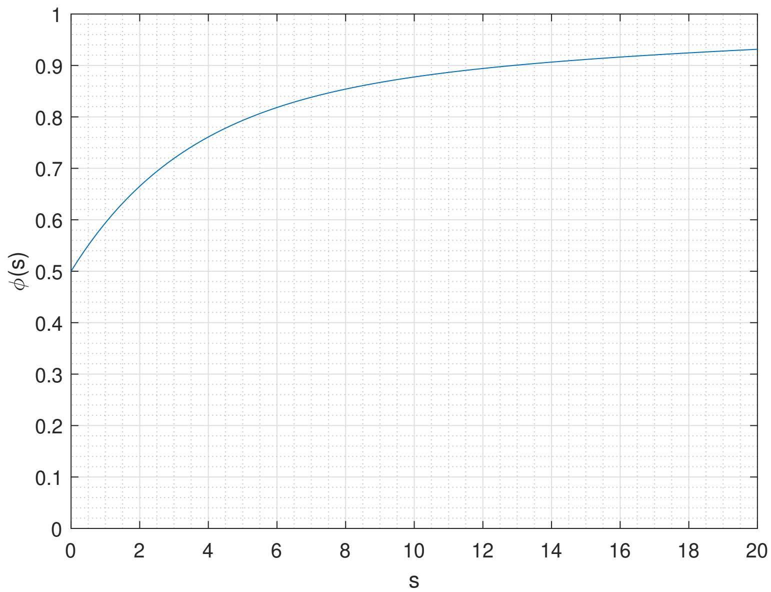


Figure 2.4: Wagner function.

it occurred in the case of the Theodorsen function, according to Wagner, a change in the angle of attack of the airfoil instantaneously generates half of the steady lift, which is not a realistic result for a dynamical system.

In contrast to Theodorsen's, Wagner function does not have an analytical expression. Nonetheless, there are some approximations, the most common being those derived by Jones [19] and Jones [20]. Both approximate the function as a sum of exponentials

$$\phi(s) \approx 1 - A_1 \mathrm{e}^{-b_1 s} - A_2 \mathrm{e}^{-b_2 s} \tag{2.10}$$

where A_i and b_i are defined constants, listed in table 2.1.

	A_1	A_2	b_1	b_2
Jones [19]	0.165	0.335	0.0455	0.3
Jones [20]	0.165	0.335	0.041	0.32

Table 2.1: Constants of the exponential approximations of the Wagner function.

2.3 State-space model

The computation of aerodynamic loads in an unsteady problem using the tools given by the classical theories is tedious and non-efficient. In order to simplify these calculations, finite-state models have been introduced to this particular area of study, due to their ability to capture the unsteady effects in a compact and effective form. Most of them are based on the classical models of Theodorsen [1] or Wagner [2]. The one introduced in this section by Taha et al. [21] proposes a new model based on the application of the Duhamel superposition principle to the Wagner equation.

As mentioned in section 2.2, Wagner [2] derived the time-response of the lift on a flat plate due to a step input. According to his model, the unsteady circulatory lift can be written as a function of the quasi-steady lift

$$l_C = l_{QS} W(s)$$

where $W(s)$ is the Wagner function, and s is the non-dimensional time, defined as a function of the free-stream velocity U and the chord of the airfoil c

$$s = \frac{2}{c} \int_0^t U(\tau) d\tau$$

Since the dynamic response of the lift for an indicial input is known, the Duhamel principle can be applied to derive the response of the lift for any input. In other words, the response due to an arbitrary excitation can be written as the superposition of the indicial time response in the first instant and the time-variation of the input variable

$$l_C(s) = \pi \rho U^2 c \left(\alpha(0) W\left(s\right) + \int_0^s \frac{d\alpha(\sigma)}{d\sigma} W\left(s - \sigma\right) d\sigma \right)$$

In this case the input is the angle of attack α, but this approach works with any other input, for example, the normal velocity $w = U \sin \alpha$, as it has been previously used in some dynamic stall models that account for high angles of attack.

In order to determine the appropriate input for the lift, it is necessary to examine the physical processes that lead to this force. According to the potential flow theory, the lift appears due to the circulation, and it is linearly dependent on it. Therefore, the quasi-steady circulation may be an appropriate aerodynamic input for this model

$$l_C(s) = \rho U(s) \left(\Gamma_{QS}(0) W(s) + \int_0^s \frac{d\Gamma_{QS}(\sigma)}{d\sigma} W(s-\sigma)\, d\sigma \right) \tag{2.11}$$

where Γ_{QS} is the quasi-steady circulation, and it is modeled to take into account the translational and the rotational effects of the airfoil

$$\Gamma_{QS} = \Gamma_{trans} + \Gamma_{rot} = \frac{1}{2} c U(t) C_{L,s}(\alpha(t)) + \pi c^2 \left(\frac{3}{4} - \hat{x}_0 \right) \dot{\alpha}(t) \tag{2.12}$$

where $C_{L,s}$ is the quasi-steady lift coefficient, and it depends directly on the angle of attack, and $\hat{x}_0$ is the position of the pitch axis normalized by the chord.

As it can be seen, this model assumes that the Wagner function can be used to represent the indicial response of the circulatory lift even at high angles of attack, which may be an invalid hypothesis. It also assumes that both the linear and nonlinear lift exhibit the same evolution over time, but building up to different values. In other words, it assumes that a nonlinear lift curve and an ideal linear curve have the same behavior, their only difference being the final value.

On the other hand, one of the problems of the Wagner function is that it does not have an analytical expression. Luckily, as it was mentioned in section 2.2, there are some good analytical approximations. In this case, it is interesting to use the one provided by Jones

[19] or Jones [20]

$$W\left(s\right)=1-A_1\mathrm{e}^{-b_1 s}-A_2\mathrm{e}^{-b_2 s} \tag{2.13}$$

Going back to the circulatory lift (equation 2.11) but rewritting it in terms of the dimensional time variable t and integrating the second term by parts

$$l_C(t)=\rho U(t)\Gamma_{eff}(t)=\rho U(t)\left(\Gamma_{QS}(t)W\left(0\right)-\int_0^t\Gamma_{QS}(\tau)\frac{dW\left(t-\tau\right)}{d\tau}d\tau\right)$$

And approximating the Wagner function with equation 2.13, the derivative of the last expression can be written as

$$\frac{dW\left(t-\tau\right)}{d\tau}=-A_i\frac{2b_i}{c}U(\tau)\mathrm{e}^{-\frac{2b_i}{c}\int_\tau^t U(\tau)d\tau},\qquad i=1,2$$

with summation on the repeated indices. Therefore, the effective circulation is written as

$$\Gamma_{eff}=\left(1-A_1-A_2\right)\Gamma_{QS}(t)+x_1(t)+x_2(t)$$

where x_i is defined as

$$x_i(t)=\int_0^t\Gamma_{QS}(\tau)A_i\frac{2b_i}{c}U(\tau)\mathrm{e}^{-\frac{2b_i}{c}\int_\tau^t U(\tau)d\tau}d\tau,\qquad i=1,2$$

which is the solution to the linear differential equation

$$\dot{x}_i=\frac{2b_iU(t)}{c}\left(-x_i(t)+A_i\Gamma_{QS}(t)\right),\qquad i=1,2 \tag{2.14}$$

with the initial condition $x_i(0)=0$. Therefore, following this model, the circulatory lift per unit span on an airfoil can be written as

$$l_C(t)=\rho U(t)\left[\left(1-A_1-A_2\right)\Gamma_{QS}(t)+x_1(t)+x_2(t)\right] \tag{2.15}$$

where Γ_{QS} is defined as in equation 2.12. In conclusion, it is possible to approximate the non-analytic expression of the circulatory lift given by Wagner to a completely defined compact two-state dynamical system. As it can be seen in equations 2.12 and 2.15 any arbitrary $C_{L,s}(\alpha)$ curve and any arbitrary motion can be plugged into the system to obtain its aerodynamic forces. No assumptions are made regarding the shape of the lift curve or the motion of the airfoil.

Chapter 3

Geometric Control

The main objective of differential geometry is to deal with concepts that are coordinate-invariant. Everything is represented in certain coordinates, but their properties are defined as intrinsic, they do not depend on the coordinate system. To introduce this philosophy of thinking, the next sections are dedicated to explain the fundamental concepts that are necessary to understand the geometric control tools that are used in this study.

The first section introduces the basic concepts of this field [22]. It mostly consists on definitions. The next section is centered around one of the most important tools in the application of differential geometry to control, the Lie brackets. These naturally lead to the definition of the symmetric products. Finally, as a more physical application of the geometric control theory, the Fliess series are introduced.

3.1 Basic concepts

Function A function $f : X \to Y$ is a map between X and Y. One point maps to just one point, it cannot map to two different points. There are different types of functions

- Injective (*one-to-one* functions): They never map different elements of the domain X onto the same element y of Y. In other words, only one element of X maps to that element y.

- Surjective (*onto* functions): For every element y in the domain Y there is at least one element x on X such that $f(x) = y$. It is not required for x to be unique, there can be more than one element of X that maps onto y.

- Bijective: It is an injective and surjective function. Each element of X maps to exactly one element of Y, and each element of Y maps to exactly one element of X. There are no unpaired elements in any of the two domains. It is possible to define an inverse function.

C^r-diffeomorphism C^r-differentiable bijection with a smooth inverse (for a unique element in X there is a unique element in Y, and the function that maps one domain to the other is smooth).

Manifold A manifold is the basic entity of differential geometry. It can be defined as a "locally flat" space (in other words, it looks like a subset of the Euclidian space) but a globally curved surface [22, 23]. For example, the Earth is globally curved, but locally it looks flat. As a counterexample, a cone is not a manifold, because the vertex neighborhood can not be "flattened" into a plane.

This definition is accompanied by the notion of two other constructions [22]. First of all, a local or coordinate chart (U, φ) is a map from a subset of the manifold to an open subset of the Euclidian space. Related to that, an atlas is defined as the collection of charts that covers the entire manifold.

Therefore, the advantage of using manifolds is revealed. If the object is in a manifold, it is still possible to represent it using different coordinate charts, but the manifold allows to focus on the properties of the object, not on the different coordinate charts.

Mathematically speaking, an n-dimensional manifold is a set that is everywhere locally diffeomorphic to the Euclidian space $\mathbb{R}^n$. In other words, it is possible to define a map that relates it with the Euclidian space.

M is an n-dimensional manifold if

1. $\forall P \in M$, there exists a neighborhood U of P in M and a diffeomorphism $\varphi : U \to \mathbb{R}$. As defined previously, the pair (U, φ) is a local chart.

2. $\cup U_i = M$. The collection $\mathcal{A} = \{(U_i, \varphi_i)\}$ is the atlas.

3. Overlap condition: $U_i \cap U_j \neq \emptyset$. Therefore, the map $\varphi_j \circ \varphi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ must be a C^r-diffeomorphism.

The space in which the manifold is embedded in (outside the manifold) is called the ambient space.

Differentiable maps In the Euclidian space, a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x if there exists

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for $x, h \in \mathbb{R}$. But this concept can not be applied in a manifold because it is not possible to add, subtract or divide.

Therefore, a function $f : M \to \mathbb{R}$ is differentiable if $\forall (U, \varphi)$ the map $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is differentiable.

In a similar way, the map $F : M \to N$ is differentiable if

1. $\psi \circ F \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable, where $\psi \circ F \circ \varphi^{-1}$ is the map between X and Y.

2. $\forall$ differentiable g on N, $g \circ F : M \to \mathbb{R}$ is differentiable.

Curve A curve γ on M is a map $\gamma : I \subset \mathbb{R} \to M$. It is differentiable if

1. $\varphi \circ \gamma : I \subset \mathbb{R} \to \mathbb{R}^n$

2. $\forall$ differentiable function f on M, $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is differentiable.

Tangent space Differential geometry is the generalization of surfaces in the Euclidian space. Therefore, the Euclidian equivalent of a tangent space would be a tangent plane, but at any point of the manifold. A more formal definition follows below.

The tangent space of M at $P \in M : T_P M$ is the set of all the velocities of the curves on M passing through the point P.

Directional derivative Assume v is a vector acting on $f \in C^\infty$. The directional derivative of f along v is defined as

$$v(f) = v \cdot f = v_1 \frac{\partial f}{\partial x_1} + \cdots + v_n \frac{\partial f}{\partial x_n} = \vec{\nabla} f \cdot \vec{v}$$

Nonetheless this definition does not work for the spaces defined by a manifold, but it introduces the concept of a vector. Looking at the definition of the directional derivative, v can be viewed as a map that takes a function f and returns a real number; in other words, $v \in T_P M \Rightarrow v : C^\infty(M) \to \mathbb{R}$.

A derivation D on M at $P \in M$ is a linear map $D : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule, $D(f \circ g) = f(P) \cdot D(g) + g(P) \cdot D(f)$, $\forall f, g \in C^\infty(M)$.

$D_P M$ is the set of all derivations at P on M.

$$v \in T_P M \Rightarrow v \in D_P(M)$$

Therefore, $T_P M \subset D_P(M)$.

Lemma: $T_P M = D_P(M)$

- Any vector $v \in T_P M$ is a derivation, taking a function $f \in C^\infty$ and giving a directional derivative of f along v.
- Any derivation $D \in D_P(M)$ comes from (corresponds to) a directional derivative along some $v \in T_P M$.

Therefore, from now on, vectors are represented in the following form

$$v = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}$$

Push forward Given a map from one manifold to another $F : M \to N$, it is possible to define a map from one tangent space to the other F_* at $P : T_P M \to T_{F(P)} N$. It is equivalent to the Jacobian in the Euclidian space

$$\dot{y} = \left[\frac{\partial F}{\partial x}\right]_P \dot{x}$$

Tangent bundle If the tangent space of a point $T_P M$ is the set of all attainable velocities at this point P, the tangent bundle TM is the set of all the attainable velocities at all points

$P \in M$.

$$TM = \coprod_{P \in M} T_P M = (P, v) | P \in M, v \in T_P M$$

Vector field A vector field $V : P \to v \in T_P M$ looks like a vector valued function on $\mathbb{R}^n$. The set of all vector fields on the manifold is denoted as $M : \mathfrak{X}(M)$. The vector field $V \in \mathfrak{X}(M)$ is a map $V : M \to TM$.

Integral curve Given $V \in \mathfrak{X}(M)$, $P \in M$, $\gamma : I \subset \mathbb{R} \to M$ with $\gamma(0) = P$:

$$\gamma'(t) = V(\gamma(t)) = V \circ \gamma(t) \qquad \forall t \in I \tag{3.1}$$

In other words, the velocity of the curve (its derivative) is equal to the velocity given by the vector field at the point on the manifold given by $\gamma(t)$, at all points.

Lemma: Given $V \in \mathfrak{X}(M)$, $P \in M$, $\exists (a, b) \ni 0$ and a $!\gamma : (a, b) \to M$ such that $\gamma(0) = P$ and γ satisfies 3.1.

If this open interval is everything $(a, b) = (-\infty, \infty)$ the vector field is complete, or equivalently, the integral curve is defined for all time. A vector field on a compact manifold is always complete.

Flow map Given $V \in \mathfrak{X}(M)$, it is possible to define the flow map $\phi_t^V : M \to M$. Physically, it can be described as the flow along a vector V for a given time t. It is a diffeomorphism.

Lie derivative In the Euclidian space, the directional derivative of a function f at the point P along the direction of V is defined as the increment of that function when P is displaced in the direction of V with respect to f evaluated in P

$$\left(\vec{\nabla} f \cdot f\right)_P = \lim_{h \to 0} \frac{f\left(P + hV\right) - f\left(P\right)}{h}$$

Nonetheless, on a manifold this definition is not valid, but leads to

$$\mathcal{L}_V f(P) = \lim_{h \to 0} \frac{f\left(\phi_h^V(P)\right) - f(P)}{h} = \left.\frac{d}{dh}\right|_{h=0} f \circ \phi_h^V(P)$$

in which the concept is similar. The derivative is still a measure of the rate of change of the function in the direction of V.

The expression of the Lie derivative is

$$\mathcal{L}_V f = V(f) = V \cdot f = \sum_{i=1}^{n} V_i \frac{\partial f}{\partial x_i}$$

As it can be seen the final expression corresponds to that of a directional derivative in the Euclidian space.

3.2 Lie Brackets

Given $X, Y \in \mathfrak{X}(M)$, $f \in C^\infty(M)$, let's compute the Lie derivative along X of the Lie derivative of f along Y

$$\mathcal{L}_X\left(\mathcal{L}_Y f\right) = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n} Y_j \frac{\partial f}{\partial x_j}\right] = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[X_i Y_j \frac{\partial^2 f}{\partial x_i x_j} + X_i \frac{\partial Y_j}{\partial x_i}\frac{\partial f}{\partial x_j}\right]$$

As it can be seen, the result of this operation is not a Lie derivative. In other words, it cannot be expressed as the Lie derivative along some vector $\mathcal{L}_V f$, such that $V \in \mathfrak{X}(M)$. A similar result is obtained for the Lie derivative along Y of the Lie derivative of f along X

$$\mathcal{L}_Y\left(\mathcal{L}_X f\right) = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n} X_j \frac{\partial f}{\partial x_j}\right] = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[Y_i X_j \frac{\partial^2 f}{\partial x_i x_j} + Y_i \frac{\partial X_j}{\partial x_i}\frac{\partial f}{\partial x_j}\right]$$

Nonetheless, when one Lie derivative is subtracted from the other

$$\mathcal{L}_X\left(\mathcal{L}_Y f\right) - \mathcal{L}_Y\left(\mathcal{L}_X f\right) = \sum_{j=1}^{n}\left[X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i}\right]\frac{\partial}{\partial x_j} f = \mathcal{L}_V f$$

the result can be expressed as the Lie derivative of f along some vector $V = [X, Y]$. This vector is called the Lie bracket, and is defined in the following way [22, 23]

$$[X, Y]_i = \frac{\partial Y_i}{\partial x_j} X_j - \frac{\partial X_i}{\partial x_j} Y_j, \qquad i \in \{1, \ldots, n\} \tag{3.2}$$

Some of the properties of the Lie bracket are easily derived from its definition. Given $f, g \in C^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$, the following properties are satisfied

- Skew-symmetry: $[X, Y] = -[Y, X]$
- Linearity: $[X + Y, Z] = [X, Z] + [Y, Z]$
- $[fX, gY] = fg\,[X, Y] + f\left(\mathcal{L}_X g\right) Y - g\left(\mathcal{L}_Y f\right) X$
- Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Physically, the Lie bracket measures the non-commutativity of two flows. The concept is represented in figure 3.1. Starting from point A, we flow along the vector field Y for some time t, in other words, we follow ϕ_t^Y and then go along the vector field X for some time s (ϕ_s^X), reaching point B. Another possible path from point A would be to flow along X for some time s and then along Y for a time t. But this second path leads to a different point B'. This occurs because the values of the X and Y depend on the point in space in which they are, and in both paths these points are different. Therefore, there are two distinct final points.

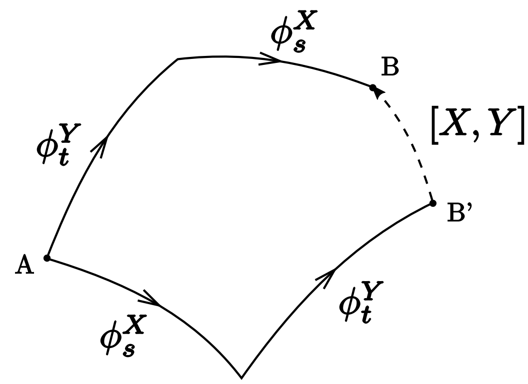


Figure 3.1: Physical representation of the Lie bracket.

Given the vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^n)$ and the point $P \in \mathbb{R}^n$, the difference between the two final points can be calculated in the following way

$$\psi(s,t;P) = \phi_t^Y \circ \phi_s^X(P) - \phi_s^X \circ \phi_t^Y(P) = \phi^Y\left(t, \phi^X(s,P)\right) - \phi^X\left(s, \phi^Y(t,P)\right)$$

Expanding ψ in two dimensions with Taylor series

$$\begin{aligned}\psi(s,t;P) = \psi(0,0;P) + s\left.\frac{\partial\psi}{\partial s}\right|_{\substack{s=0\\t=0}} + t\left.\frac{\partial\psi}{\partial t}\right|_{\substack{s=0\\t=0}} + \frac{s^2}{2}\left.\frac{\partial^2\psi}{\partial s^2}\right|_{\substack{s=0\\t=0}} + \frac{t^2}{2}\left.\frac{\partial^2\psi}{\partial t^2}\right|_{\substack{s=0\\t=0}} \\ + st\left.\frac{\partial^2\psi}{\partial s\partial t}\right|_{\substack{s=0\\t=0}} + O\left(s^3, t^3\right)\end{aligned}$$

The first derivative of the function ψ is

$$\begin{aligned}\left.\frac{\partial\psi}{\partial s}\right|_{\substack{s=0\\t=0}} &= \left(\phi_t^Y\right)_* \cdot X\left(\phi_s^X(P)\right)\Big|_{\substack{s=0\\t=0}} - X\left(\phi_s^X \circ \phi_t^Y(P)\right)\Big|_{\substack{s=0\\t=0}} \\ &= \left(\phi_t^Y\right)_* \cdot X(P)\Big|_{\substack{s=0\\t=0}} - X\left(\phi_s^{1X}(P)\right)\Big|_{\substack{s=0\\t=0}} = X(P)\Big|_{\substack{s=0\\t=0}} - X(P)\Big|_{\substack{s=0\\t=0}} = 0\end{aligned}$$

The evaluation of the flow at a time equal to 0 is the point at which it is evaluated. Similarly, the Jacobian $\left(\phi_t^Y\right)_*$ at a time 0 is the identity matrix. Therefore, all the terms in ψ vanish except the last one (and the higher order terms).

$$\psi(s,t;P) = st\frac{\partial^2\psi}{\partial s\partial t}\bigg|_{\substack{s=0\\t=0}} + O\left(s^3,t^3\right) = st\left[X,Y\right] + O\left(s^3,t^3\right)$$

In conclusion, the difference between both paths is in the direction of the Lie bracket. Therefore, if $[X,Y]=0$ for all $P\in M$ the vector fields X,Y are commutative, meaning that the final point is the same for both paths.

This interpretation leads to another representation of the Lie bracket. It can also be defined as the derivative of the curve $\gamma : I\subset\mathbb{R}\to\mathbb{R}^n$ at the origin, in the form $\gamma'(0)=[X,Y]$, where γ is a differentiable curve defined as [22]

$$\gamma(t) = \phi^{-Y}_{\sqrt{t}}\circ\phi^{-X}_{\sqrt{t}}\circ\phi^{Y}_{\sqrt{t}}\circ\phi^{X}_{\sqrt{t}}(P)$$

But the depth of this perception of the Lie bracket as a measure of non-commutativity can only be truly appreciated when it is applied to control theory. It has several implications, especially regarding system controllability.

Assume the following nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t)\right) + \sum_{j=1}^{m}\mathbf{g}_j\left(\mathbf{x}(t)\right)u_j(t) \tag{3.3}$$

where $\mathbf{x}$ is the state vector on an n-dimensional manifold M^n, f is the drift vector field (the dynamics of the system that cannot be controlled), and $\mathbf{g}_j$'s are the control vector fields with the corresponding inputs u_j's.

In a driftless system ($\mathbf{f}=\mathbf{0}$) it is possible to generate motion along a control vector field $\mathbf{g}_k$ by turning off all the control inputs except for u_k. However, according to what has been discussed in this section, it is possible to move along additional and non-intuitive directions determined by the Lie bracket

$$\left[\mathbf{g}_j, \mathbf{g}_k\right] = \frac{\partial \mathbf{g}_k}{\partial \mathbf{x}}\mathbf{g}_j - \frac{\partial \mathbf{g}_j}{\partial \mathbf{x}}\mathbf{g}_k \tag{3.4}$$

If the resulting vector field $\left[\mathbf{g}_j, \mathbf{g}_k\right]$ is linearly independent to the ones that generated it, $\mathbf{g}_j$ and $\mathbf{g}_k$, then it is possible to generate motion in an unactuated direction. In other words, through the adequate manipulation of the control inputs u_j and u_k, one can generate motion along a new direction. For example, a possible control law would be out-of-phase periodic signals of the corresponding inputs [24, 25, 26].

Example 3.1. *On $M = \mathbb{R}^3$ consider the vector fields $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$. The resulting Lie bracket is $[X, Y] = \frac{\partial}{\partial z}$. In other words, if we flow along X and then along Y, and then along $-X$ and finally along $-Y$ for the same amount of time, the net result is a displacement in the z-direction. Therefore, the addition of the Lie bracket allows to control a system in a direction previously deemed as uncontrollable.*

Consequently, the Lie bracket provides a way to generate motion in a previously unactuated direction, one in which there is no direct control authority. Essentially, it gives a new direction of actuation, one that cannot be obtained by the classical linear control theory. Therefore, a system that is deemed as uncontrollable in linear control theory may be controllable under geometric nonlinear control.

3.3 Symmetric products

The symmetric product is computed as an iteration of Lie brackets in the following form

$$< X : Y >=< Y : X >= [X, [f, Y]] = [Y, [f, X]] \tag{3.5}$$

It appears as a mechanism to generate forces in additional directions using high-frequency oscillatory control inputs. In geometric control a zero-mean oscillation input can lead to net motion in a certain direction [16, 27]. Combining geometric control and averaging theory [28, 29] with the chronological calculus tools developed by Agračev and Gamkrelidze [30], it can be proved that the net effect of an oscillatory input on the averaged dynamics of the system 3.3 is given by the vector field $\left[\mathbf{g}_j, \left[f, \mathbf{g}_j\right]\right]$ [31].

Therefore, the effect of the symmetric product does not go unnoticed in the overall dynamics of the system. Its effects can be realized by high-amplitude high-frequency oscillatory inputs. For example, assuming an input of the form $u_j = \omega U_j \cos \omega t$ with high enough frequency, the averaged dynamics of 3.3 are given by [31]

$$\dot{\bar{\mathbf{x}}} = \mathbf{f}\left(\bar{\mathbf{x}}\right) - \sum_{j,k=1}^{m} \frac{U_j U_k}{4} < \mathbf{g}_j : \mathbf{g}_k > \left(\bar{\mathbf{x}}\right) \tag{3.6}$$

where the over-bar indicates an averaged value.

The structure of the product $< \mathbf{g}_j : \mathbf{g}_k >$ can be different than that of the vector fields that generate it. For example, the control vector field $\mathbf{g}_j$ may have a zero component in some direction, meaning that there is no authority on that direction. But the same component of the symmetric product may not cancel, meaning that it is possible to generate a new force in that direction through the periodic oscillation of the control input u_j. At the same time, the control vector field may have some nonzero components that vanish in $< \mathbf{g}_j : \mathbf{g}_k >$, indicating the canceling of the instantaneous effects of $\mathbf{g}_j$ over a cycle of oscillation.

It is this type of behavior that is exploited in this study. The main objective is to identify the effects of oscillatory motion on a two-dimensional airfoil, to see if there is an overall increase in its aerodynamic properties. To do so, the airfoil is modeled as a nonlinear control system, as it is further explained in sections 2.3 and 4.

3.4 Fliess functional expansion

A control tool that is extremely useful in the study of dynamical systems is an input-output representation. It allows a direct evaluation of the system without the need of solving the entire dynamical system.

Instead of solving the states of the system for a given set of inputs and then obtaining the output, an input-output representation is a direct connection between the inputs and the outputs. In other words, the states of the systems are eliminated, with the need of recalculating the entire system for every new set of inputs.

For example, in a linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

the input-output response has the following form

$$y(t) = C\left(e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right) \tag{3.7}$$

Nonetheless, the definition of an input-output response such as 3.7 for a nonlinear system is not as intuitive. First of all, it is necessary to denote [32]

$$\mathbf{g}_0 = \mathbf{f}, \qquad u_0 = 1$$

Secondly, the following set of multi-integrals should be defined [33, 32]

$$\begin{aligned}\int_0^t d\xi_i &= \int_0^t u_i(\tau)d\tau \\ \int_0^t d\xi_{i_{k+1}} d\xi_{i_k} \ldots d\xi_{i_1} &= \int_0^t u_{i_{k+1}}(\tau) \int_0^\tau d\xi_{i_k} \ldots d\xi_{i_1}, \quad k \leq 1\end{aligned} \tag{3.8}$$

Then, the input-output response of a nonlinear system 3.3 can be expressed as the series expansion

$$y_j(t) = h_j(x_0) + \sum_{k=0}^{\infty} \sum_{i_k,\ldots,i_0=0}^{m} \mathcal{L}_{\boldsymbol{g_{i_0}}} \ldots \mathcal{L}_{\boldsymbol{g_{i_k}}} h_j(x_0) \int_0^t d\xi_{i_k} \ldots d\xi_{i_0} \tag{3.9}$$

which is the Fliess functional expansion of the system [33, 32].

Chapter 4

System

4.1 Definition

The purpose of this work is to obtain a Reduced Order Model (ROM) of a two-dimensional pitching and plunging airfoil like the one in figure 4.1. The system proposed in the present study derives from the state space model given by Taha et al. [21] introduced in section 2.3. The purpose of this system is to model the unsteady flow, an infinite-dimensional space, using a finite-dimensional state space model. In this case, the resulting state space is composed of two aerodynamic states.

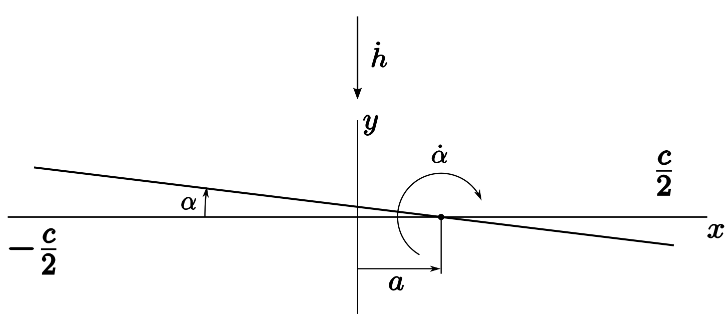


Figure 4.1: Graphical representation of the two-dimensional airfoil in a pitching and plunging motion.

According to Taha et al. [21], the dynamics of a wing flapping with an angle of attack $\alpha(t)$ can be approximated with the two differential equations

$$\dot{x}_i = \frac{2b_i U(t)}{c}\left(-x_i(t) + A_i \Gamma_0(t)\right), \qquad i = 1, 2 \tag{4.1}$$

where U is the free-stream velocity, c is the chord of the airfoil, Γ_0 is the quasi-steady circulation, and A_i and b_i are known constants defined in section 2.2.

However, since the addition or deletion of a state does not change the behavior of the system or the calculations, as a matter of practicality, the system is reduced to one equation. The differential equations given by 4.1 are substituted by a one state approximation

$$\dot{x}_c = -\frac{1}{\tau_c} x_c + \frac{1}{2}\Gamma_0\left(\alpha, \dot{\alpha}, \dot{h}\right) \tag{4.2}$$

where x_c is the internal aerodynamic state that represents the dynamics of the circulatory lift component and $\tau_c = \frac{2bU}{c}$ is its time constant.

The input of this system is the quasi-steady circulation. However, the model given by Taha et al. [21] does not account for plunging, so equation 2.12 has to be modified in the following form

$$\Gamma_0 = \frac{1}{2} c U C_{L,s}\left(\alpha + \arctan\frac{\dot{h}}{U}\right) + \pi c^2\left(\frac{3}{4} - \hat{x}_0\right)\dot{\alpha}$$

where the circulation depends on the quasi-steady lift $C_{L,s}$ and the pitching position normalized by the chord $\hat{x}_0 = \frac{a}{c}$. The expression of the quasi-steady lift depends on the airfoil section, and it is a nonlinear function of the effective angle of attack, which depends on the actual angle α and the plunging velocity $\dot{h}$. So, ultimately, these are the real inputs of the system, the motion of the wing.

The output of the system is the total lift, which is split into circulatory and non-circulatory components, the former given by the equation

$$l_C(t) = \rho U(t) \left[(1 - A_1 - A_2) \Gamma_0(t) + x_1(t) + x_2(t) \right]$$

where ρ is the density of the surrounding fluid.

For a pitching and plunging motion, the non-circulatory term is given by the instant acceleration of the airfoil in the following form

$$l_{NC}(t) = m_v \left[\left(U \cos\alpha - \dot{h} \sin\alpha \right) \dot{\alpha} + \ddot{h} \cos\alpha - \ddot{\alpha} a \right] \tag{4.3}$$

where $m_v = \frac{\pi}{4} \rho c^2$ is the apparent mass.

Therefore, the output $l = l_C(t) + l_{NC}(t)$ depends on the plunging motion of the wing h as well as on the angle of attack α. However, some of the variables in equation 4.3 are their first and second derivatives. To avoid this complication, the accelerations $\ddot{\alpha}$ and $\ddot{h}$ are defined as the inputs of the system, while the velocities and the original variables become states.

Then the output is function of the states but also of the input. Nonetheless, in an input-output representation it is preferable to have a direct dependence of the states, not the input. Luckily, according to Taha and Rezaei [34], the effect of viscosity in the unsteady motion of the airfoil is a lag. And, most importantly, this lag acts on the non-circulatory component of the lift.

Therefore, the lag due to viscosity can be modeled in a similar way to that of the circulatory lift. Thus the viscous state x_v is defined. On the other hand, at the same time, in the non-circulatory lift (equation 4.3) the two last terms depend on the inputs of the system. To avoid it, the new state x_v is defined in such a way that these inputs do not appear on the output lift equation. The new expression of the non-circulatory lift becomes

$$l_{NC} = m_v \left[\left(U\cos\alpha - \dot{h}\sin\alpha \right) \dot{\alpha} + x_v \cos\alpha \right]$$

and the differential equation that governs the dynamics of this new viscous state is similar to that of the circulatory state

$$\dot{x}_v = -\frac{1}{\tau_v} - \frac{\ddot{\alpha} a}{\tau_v} + \frac{\ddot{h}\cos\alpha}{\tau_v}$$

where τ_v is the time constant of the dynamics that govern the aerodynamic state of the non-circulatory lift. This dynamical behavior only appears when the viscosity is taken into account, causing a lag in the generation of non-circulatory lift. In other words, this equation does not appear in potential flow.

Therefore, the final set of equations of the system of study is

$$\frac{d}{dt}\begin{pmatrix} x_c \\ x_v \\ \alpha \\ \dot{\alpha} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\tau_c}x_c + \frac{1}{2\tau_c}\Gamma_0\left(\alpha, \dot{\alpha}, \dot{h}\right) \\ -\frac{1}{\tau_v}x_v \\ \dot{\alpha} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{a}{\tau_v} \\ 0 \\ 1 \\ 0 \end{pmatrix} u_\alpha(t) + \begin{pmatrix} 0 \\ \frac{\cos\alpha}{\tau_v} \\ 0 \\ 0 \\ 1 \end{pmatrix} u_h(t) \tag{4.4}$$

where the inputs are the accelerations $u_\alpha = \ddot{\alpha}$ and $u_h = \ddot{h}$, and the quasi-steady circulation is expressed in the following way

$$\Gamma_0 = k_\alpha C_{L,s}\left(\alpha + \arctan\frac{\dot{h}}{U}\right) + k_{\dot{\alpha}}\dot{\alpha} \tag{4.5}$$

where $k_\alpha = \frac{1}{2}cU$ and $k_{\dot{\alpha}} = \pi c^2\left(\frac{3}{4} - \hat{x}_0\right)$ are constants that represent the translatory and rotational contributions of the circulation.

4.2 Lift

The output of the system derived in the previous section is the lift of the airfoil, which is the addition of the circulatory and the non-circulatory contributions

$$L = H(x) = \rho U \left(x_c + \frac{1}{2}\Gamma_0(\alpha, \dot{\alpha}, \dot{h}) \right) + m_v \left[\left(U \cos\alpha - \dot{h} \sin\alpha \right) \dot{\alpha} + x_v \cos\alpha \right] \tag{4.6}$$

4.2.1 Lie derivatives

Once the unsteady ROM is settled, it is possible to perform some preliminary geometric control analysis. For example, the symmetric products between the vector fields $\mathbf{g}_\alpha$ and $\mathbf{g}_h$ can determine if there is some type of symmetry breaking due to high-frequency oscillations [8]. However, this phenomenon will only occur if these symmetric product vectors act on the direction of the lift.

Therefore, in order to study this nonlinear effect it is necessary to project these symmetric products on the gradient of the lift. This operation is the Lie derivative of the lift along these vectors. It represents the rate of change of the lift along the dynamics given by the symmetric product.

$$\mathcal{L}_{\langle \boldsymbol{g_\alpha} : \boldsymbol{g_\alpha} \rangle} H = 0$$

$$\mathcal{L}_{\langle \boldsymbol{g_\alpha} : \boldsymbol{g_h} \rangle} H = -\frac{m_v \sin(2\alpha)}{2\tau_v}$$

$$\mathcal{L}_{\langle \boldsymbol{g_h} : \boldsymbol{g_h} \rangle} H = -\frac{\rho U k_\alpha}{2\tau_c} \frac{\partial^2 C_{L,s} \left(\alpha + \arctan \frac{\dot{h}}{U} \right)}{\partial \dot{h}^2}$$

Evaluating the results at the equilibrium point given by the averaged dynamics ($x_c = \Gamma_0/2$, $\dot{\alpha} = \dot{h} = x_v = 0$), the following expressions are obtained

$$\mathcal{L}_{\langle \boldsymbol{g_\alpha} : \boldsymbol{g_\alpha} \rangle} H = 0 \tag{4.7}$$

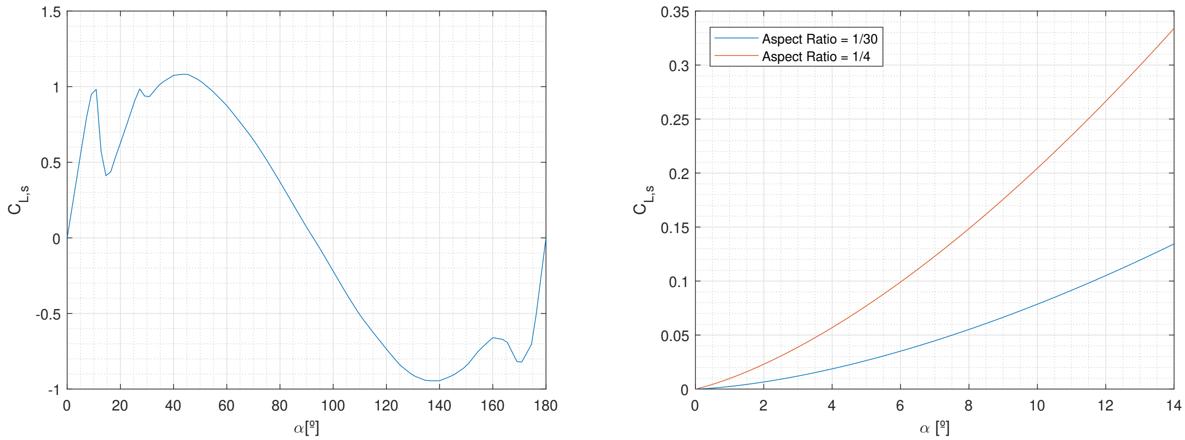
$$\mathcal{L}_{\langle \boldsymbol{g_\alpha} : \boldsymbol{g_h} \rangle} H = -\frac{m_v \sin(2\alpha)}{2\tau_v} \tag{4.8}$$

$$\mathcal{L}_{\langle \boldsymbol{g_h} : \boldsymbol{g_h} \rangle} H = -\frac{\rho k_\alpha}{2U\tau_c} \frac{\partial^2 C_{L,s}(\alpha)}{\partial \alpha^2} \tag{4.9}$$

As it can be seen, the effect of pitching is not significant on the lift, since the Lie derivative is equal to zero. However, the interaction between pitching and plunging generates an additional lift force through the non-circulatory term. This enhancement is proportional to the virtual mass and inversely proportional to the viscosity time constant. It attains its maximum value at an angle of attack of 45°.

Nonetheless, the most important non-linear contribution is given by the high-frequency heaving oscillations alone. They generate an additional lift force that depends on the second derivative of the steady lift coefficient with respect to the angle of attack. Looking at the steady lift curve of a conventional airfoil, it can be seen that $C''_{L,s} = 0$ for low angles of attack. In other words, in the linear regime in which most airplanes flow there is no enhancement effect. However, in the post-stall regime, where $C''_{L,s} > 0$, the plunging motion provides this additional force.

On the contrary, on a delta wing the lift curve with respect to the angle of attack is nonlinear in its whole range due to an attached leading edge vortex or LEV. As a result, $C''_{L,s} > 0$ in the pre-stall regime, meaning that we always have this additional lift force, even at low or zero angle of attack.



(a) NACA 0012, $Re = 500,000$ [35].

(b) Delta wing [36].

Figure 4.2: Steady lift variation of a two-dimensional NACA 0012 airfoil [35] and a three-dimensional delta wing [36].

4.2.2 Fliess functional expansion

Another way to study the effect that the unsteady motion has on the wing is the Fliess functional expansion introduced in section 3.4. These series provide a more direct way to understand the effects of the unsteady motion in the airfoil.

Since the purpose of this thesis is to study the nonlinearities a first-order expansion is not enough. Therefore, in this section and on the following ones the expansions are going to be of second order, to see the most important nonlinear effects. Third and higher order series are not studied to simplify the analysis of the results, but they could easily be performed using the same methodology.

Following the procedure explained in section 3.4, the second order Fliess functional expansion is

$$
\begin{aligned}
H(t) \approx \rho U \Gamma_0 + \left[\frac{\rho U k_{\dot{\alpha}}}{2} + m_v \cos\alpha \left(U - \frac{e}{\tau_v}\right)\right] \int_0^t u_\alpha(\tau) d\tau \\
+ \left[\frac{m_v \cos^2\alpha}{\tau_v} + \frac{\rho k_\alpha}{2} \frac{\partial C_{L,s}}{\partial \alpha}\right] \int_0^t u_h(\tau) d\tau \\
+ \left[\frac{\rho U}{2}\left(\frac{k_{\dot{\alpha}}}{\tau_c} + k_\alpha \frac{\partial C_{L,s}}{\partial \alpha}\right) + \frac{m_v e \cos\alpha}{\tau_v^2}\right] \int_0^t d\tau_1 \int_0^{\tau_1} u_\alpha(\tau_2) d\tau_2 \\
+ \left[\frac{\rho k_\alpha}{2} \frac{\partial C_{L,s}}{\partial \alpha} - \frac{m_v \cos^2\alpha}{\tau_v^2}\right] \int_0^t d\tau_1 \int_0^{\tau_1} u_h(\tau_2) d\tau_2 - m_v \sin\alpha \int_0^t u_\alpha(\tau_1) d\tau_1 \int_0^{\tau_1} u_h(\tau_2) d\tau_2 \\
- m_v \sin\alpha \int_0^t u_h(\tau_1) d\tau_1 \int_0^{\tau_1} u_\alpha(\tau_2) d\tau_2 + \frac{\rho k_\alpha}{2U} \frac{\partial^2 C_{L,s}}{\partial \alpha^2} \int_0^t u_h(\tau_1) d\tau_1 \int_0^{\tau_1} u_h(\tau_2) d\tau_2
\end{aligned}
$$

As it can be seen, the Fliess expansion depends on the inputs of the system. The function u_α represents the pitching motion of the wing, and u_h the plunging. In this work, it is assumed that both inputs are sinusoidal with the same frequency, and with a phase shift φ between them

$$
\begin{aligned}
u_\alpha(t) &= \omega U_\alpha \cos(\omega t) \\
u_h(t) &= \omega U_h \cos(\omega t + \varphi)
\end{aligned}
\tag{4.10}
$$

where U_α and U_h are the amplitudes of the pitching and plunging motions respectively, and ω is the angular velocity of the movement.

Introducing these expression in the expansion, the following equation is obtained

$$\begin{aligned}
H(t) \approx \rho U \Gamma_0 &+ \left[\frac{\rho U k_{\dot{\alpha}}}{2} + m_v \cos\alpha \left(U - \frac{e}{\tau_v}\right)\right] U_\alpha \sin(\omega t) \\
&+ \left[\frac{m_v \cos^2\alpha}{\tau_v} + \frac{\rho k_\alpha}{2}\frac{\partial C_{L,s}}{\partial \alpha}\right] U_h \left[\sin(\omega t + \varphi) - \sin\varphi\right] \\
&+ \left[\frac{\rho U}{2}\left(\frac{k_{\dot{\alpha}}}{\tau_c} + k_\alpha \frac{\partial C_{L,s}}{\partial \alpha}\right) + \frac{m_v e \cos\alpha}{\tau_v^2}\right] \frac{U_\alpha}{\omega}\left[1 - \cos(\omega t)\right] \\
&+ \left[\frac{\rho k_\alpha}{2}\frac{\partial C_{L,s}}{\partial \alpha} - \frac{m_v \cos^2\alpha}{\tau_v^2}\right] \frac{U_h}{\omega}\left[\cos\varphi - \omega t \sin\varphi - \cos(\omega t + \varphi)\right] \\
&- m_v \sin\alpha \frac{U_\alpha U_h}{4}\left[2\sin\varphi\left(\omega t - 2\sin(\omega t)\right) - \cos(2\omega t + \varphi) + \cos\varphi\right] \\
&- m_v \sin\alpha \frac{U_\alpha U_h}{4}\left[\cos\varphi - 2\omega t \sin\varphi - \cos(2\omega t + \varphi)\right] \\
&+ \frac{\rho k_\alpha}{2U}\frac{\partial^2 C_{L,s}}{\partial \alpha^2}\frac{U_h^2}{2}\left[\sin\varphi - \sin(\omega t + \varphi)\right]^2
\end{aligned}$$

It is possible to express the lift as the steady value plus an increment that varies over time, $H(t) = H_0 + \Delta H(t)$. This unsteady variation is the variable of interest in this case. However, the expression is still too complex and does not provide the information that we are interested in. As it has been stated several times, the purpose of this study is to evaluate the overall effect of high-frequency oscillations in pitching and plunging. In other words, the main objective is to obtain the average result of the oscillations, to see if there is a real enhancement of the lift. Therefore, it is more interesting to look at the average of the lift increment over one period of time ($T = 2\pi/\omega$)

$$\begin{aligned}
\Delta \bar{H} \approx &- \left[\frac{m_v \cos^2\alpha}{\tau_v} + \frac{\rho k_\alpha}{2}\frac{\partial C_{L,s}}{\partial \alpha}\right] U_h \sin\varphi \\
&+ \left[\frac{\rho U}{2}\left(\frac{k_{\dot{\alpha}}}{\tau_c} + k_\alpha \frac{\partial C_{L,s}}{\partial \alpha}\right) + \frac{m_v e \cos\alpha}{\tau_v^2}\right] \frac{U_\alpha}{\omega} \\
&+ \left[\frac{\rho k_\alpha}{2}\frac{\partial C_{L,s}}{\partial \alpha} - \frac{m_v \cos^2\alpha}{\tau_v^2}\right] \frac{U_h}{\omega}\left(\cos\varphi - \pi \sin\varphi\right) \\
&- m_v \sin\alpha \frac{U_\alpha U_h}{2}\cos\varphi + \frac{\rho k_\alpha}{2U}\frac{\partial^2 C_{L,s}}{\partial \alpha^2}\frac{U_h^2}{2}\left[1 - \frac{1}{2}\cos(2\varphi)\right]
\end{aligned} \tag{4.11}$$

Since the focus is in high-frequency oscillations, some of the terms of the previous equation can be neglected to focus on the ones that contribute more to the lift enhancement. Therefore, the average increment in lift can be approximated to

$$\Delta\bar{H} \sim -\left[\frac{m_v \cos^2\alpha}{\tau_v} + \frac{\rho k_\alpha}{2}\frac{\partial C_{L,s}}{\partial\alpha}\right] U_h \sin\varphi \\ - m_v \sin\alpha \frac{U_\alpha U_h}{2}\cos\varphi + \frac{\rho k_\alpha}{2U}\frac{\partial^2 C_{L,s}}{\partial\alpha^2}\frac{U_h^2}{2}\left[1 - \frac{1}{2}\cos(2\varphi)\right] \quad (4.12)$$

These terms are very similar to the results obtained in section 4.2.1 in the calculation of the Lie derivatives of the lift along the symmetric products $\langle \boldsymbol{g_\alpha} : \boldsymbol{g_h} \rangle$ and $\langle \boldsymbol{g_h} : \boldsymbol{g_h} \rangle$.

According to this simplification, when the pitching and plunging motions are in phase, the added mass term that appears due to the interaction between them through the non-circulatory lift decreases the total value of the force. Nonetheless, in contrast, high-frequency plunging oscillations generate an increase in the lift that is proportional to the second derivative of the steady lift coefficient with respect to the angle of attack. As it has been stated in section 4.2.1, in conventional wings, this derivative is zero (due to the linear regime); however, in the post-stall regime or in delta wings the derivative is positive. Therefore, in these cases, there is a positive increase in the lift when the airfoil follows a plunging motion.

Regarding the effect of the phase shift, it is notable the first order term that appears if there is a phase difference between the pitching and plunging motions. This term combines the effects of the non-circulatory lift through the added mass and the circulatory contribution through the first derivative of the steady lift coefficient with the angle of attack. Contrary to the second order plunging term, this expression is also non-zero and positive in the linear regime of conventional wings. However, outside of the linear range, for certain angles of attack, the first derivative can be negative, as it can be seen in figure 4.2a. Consequently, this first order term will increase or decrease the lift depending on the value of the phase φ but also on the mean angle of attack α.

As for the second order terms, when the phase shift is negative (the pitching motion is delayed with respect to the plunging), the interaction between both motions through the non-circulatory lift causes an increase in the lift, contrary to the case in which they are in phase. On the other hand, the last term, the one due to the second order effects of the plunging motion, is positive regarding the phase shift.

Finally, analyzing the results, it is clear that the enhancement in the lift reaches its maximum value when the phase shift is $\varphi = -90°$. In other words, the maximum increase in the lift occurs when the pitching motion is delayed 90° with respect to the plunging.

4.3 Drag

The other aerodynamic force that has to be taken into account (and probably the second most important characteristic of an aerodynamic system) is the drag. In inviscid flows the only force that appears on an airfoil is the lift, perpendicular to the surface. Nonetheless, viscous and unsteady flows can have force in the direction parallel to the free stream velocity: the drag force. It has two components, the one due to pressure and the one due to the leading edge suction.

Regarding the pressure contribution, as it occurs with the lift, a large portion of the drag comes from the pressure forces normal to the wing surface. Thus, this drag is the component of these pressure forces that is parallel to the free-stream velocity. It can be expressed in terms of the lift

$$D_{pressure} = \frac{L}{\cos\alpha}\sin\alpha = L\tan\alpha$$

Nonetheless, this is not the only contribution on the drag. As Garrick [37] mentioned, the suction force should be considered. It appears due to the high vorticity that occurs on the leading edge [38, 39]. Assuming a rounded leading edge, like in the case of an airfoil, when the wing is at a certain angle of attack, the stagnation point moves away from the leading edge to a point in the lower surface. Thus, when the flow reaches the airfoil, it stops at the stagnation point and then travels around the leading edge toward the upper surface. In consequence, the velocity at the leading edge is high. However, if instead of an airfoil we have a flat plate, the thickness in the leading edge decreases to zero. Therefore, the flow has to turn 180° at the leading edge. This turn requires infinite acceleration and, consequently, infinite velocity. According to Bernoulli's equation, the pressure at the leading edge becomes extremely low, compared to that at infinity, resulting in a suction force.

It is modeled with the following formula [37]

$$F_S = -\pi\rho\frac{c}{2}S^2 \tag{4.13}$$

where S accounts for the leading edge suction

$$S = \frac{\sqrt{2}}{2}\left[2C(k)Q - \frac{c}{2}\dot{\alpha}\right] \tag{4.14}$$

where Q is the velocity normal to the airfoil at the three-quarter chord point (positive downward)

$$Q = U\sin\alpha + \dot{h}\cos\alpha + \left(\frac{c}{4} - e\right)\dot{\alpha}$$

and $C(k)$ is the Theodorsen function, a complex function introduced in section 2.1 that only depends on the reduced frequency $k = \frac{\omega c}{2U}$. It models the time delay in the lift due to the change in motion of the wing.

Looking at equation 4.13, what stands out the most is the negative sign. The suction force acts as a thrust force, in the direction of motion. In other words, if it is bigger than the pressure drag, the total force in the direction parallel to the free stream velocity will point forwards, becoming a propulsive force instead of a drag force.

Nonetheless, S depends on the Theodorsen function $C(k)$, which is in the frequency domain, whereas we are interested in working in the time domain. Thus, the function has to be written as a function of the current states of the developed system.

According to Theodorsen [1], the circulatory lift is

$$L_C = \rho U c \pi C(k) Q$$

And in the current model [8]

$$L_C = \rho U \left(x_c + \frac{1}{2} \Gamma_0 \left(\alpha, \dot{\alpha}, \dot{h} \right) \right)$$

Since both methods are used in the modeling of a pitching-plunging airfoil they should be equivalent. So, equating both expressions, the following relation is obtained

$$C(k) Q = \frac{1}{\pi c} \left(x_c + \frac{1}{2} \Gamma_0 \left(\alpha, \dot{\alpha}, \dot{h} \right) \right)$$

Consequently, suction (equation 4.14) can be written as

$$S = \frac{\sqrt{2}}{2} \left[\frac{2}{\pi c} \left(x_c + \frac{1}{2} \Gamma_0 \left(\alpha, \dot{\alpha}, \dot{h} \right) \right) - \frac{c}{2} \dot{\alpha} \right]$$

Finally, the total drag of the wing is the addition of the pressure drag and the suction force. Accordingly, the drag of the airfoil, written as a function of the states of the system is

$$\begin{aligned} D = \left\{ \rho U \left(x_c + \frac{1}{2} \Gamma_0(\alpha, \dot{\alpha}, \dot{h}) \right) + m_v \left[\left(U \cos\alpha - \dot{h} \sin\alpha \right) \dot{\alpha} + x_v \cos\alpha \right] \right\} \tan\alpha \\ - \frac{\pi \rho c}{4} \left[\frac{c\dot{\alpha}}{2} - \frac{1}{\pi c} \left(2x_c + k_{\dot{\alpha}} \dot{\alpha} + k_\alpha C_{L,s} \right) \right]^2 \end{aligned} \tag{4.15}$$

4.3.1 Lie derivatives

As it was previously done for the lift, in order to compute the rate of change of the drag force due to the different motions, the Lie brackets along the symmetric products are calculated. Evaluating the expressions at the equilibrium point given by the averaged dynamics

$$\mathcal{L}_{\langle \boldsymbol{g_\alpha} : \boldsymbol{g_\alpha} \rangle} D = 0 \tag{4.16}$$

$$\mathcal{L}_{\langle \boldsymbol{g_\alpha} : \boldsymbol{g_h} \rangle} D = -\frac{m_v \sin^2 \alpha}{\tau_v} \tag{4.17}$$

$$\mathcal{L}_{\langle \boldsymbol{g_h} : \boldsymbol{g_h} \rangle} D = \frac{\rho k_\alpha}{2U^2 \tau_c} \frac{\partial^2 C_{L,s}}{\partial \alpha^2} \left(\frac{2\Gamma_0}{\pi c} - U \tan \alpha \right) \tag{4.18}$$

As it happened in the case of the lift force, the pitching motion does not affect the drag. However, the interaction between pitching and plunging generates an increase in drag that is proportional to the added mass. So, as it occurred with the lift, this contribution appears due to the non-circulatory term. In fact, this Lie derivative is equal to its counterpart for the lift force but multiplied by $\tan \alpha$. Therefore, it can be stated that this Lie derivative comes from the pressure drag, not from the suction force.

Again, the most significant contribution is the one due to the plunging alone. It is proportional to the second derivative of the static lift coefficient with respect to the angle of attack, so the same reasoning that was developed in section 4.2.1 can be applied to this derivative. For conventional airfoils, since the lift curve is linear in the pre-stall regime, this contribution will be non-zero only for angles of attack higher than the stall angle. Nonetheless, for delta wings, due to the curvature of their lift curve, it will be positive in the whole range of angles of attack.

In fact, one of the terms of the Lie derivative is equal to that of the lift derivative but multiplied by $\tan \alpha$. Consequently, this term comes from the pressure drag. However, contrary to what happened for the lift, it increases with the angle of attack until it reaches its maximum at $\alpha = 90°$. Therefore, for low angles of attack this contribution will be almost null.

Nevertheless, there is another term in the Lie derivative that comes from the suction force. It is proportional to the value of the quasi-steady circulation and, in turn to the lift coefficient. In fact, in equilibrium conditions, the term inside the brackets is $U\left(\frac{C_{L,s}}{\pi} - \tan\alpha\right)$. Therefore, depending on the angle of attack this contribution may be positive or negative.

In this Lie derivative, both contributions appear to be opposite; while the former seems to increase the drag, the latter tends to decrease it. They follow the expected behavior of the pressure drag and the suction force respectively. However, for some angles of attack, the plunging contribution may be negative, increasing the drag of the airfoil. Therefore, in some cases, it may be possible for the suction force to overcome the drag, generating a net force forwards.

4.3.2 Fliess functional expansion

A more direct way to see the effects of the pitching and plunging motions on the drag force is to compute the Fliess functional expansion introduced in section 3.4. These series approximate the output of a dynamical system as a function of the inputs. In order to study the nonlinear response of the system, the second order Fliess expansion of the drag is computed.

$$
\begin{aligned}
D(t) \approx\ & \rho U \Gamma_0 \tan\alpha - \frac{\rho \Gamma_0^2}{\pi c} \\
& + \left[\frac{\rho U k_{\dot{\alpha}} \tan\alpha}{2} + m_v \sin\alpha \left(U - \frac{e}{\tau_v} \right) - \frac{\rho \Gamma_0}{2\pi c} \left(2k_{\dot{\alpha}} - \pi c^2 \right) \right] \int_0^t u_\alpha(\tau) d\tau \\
& + \left[\frac{m_v \sin(2\alpha)}{2\tau_v} + \frac{\rho k_\alpha}{2U} \frac{\partial C_{L,s}}{\partial \alpha} \left(U \tan\alpha - \frac{2\Gamma_0}{\pi c} \right) \right] \int_0^t u_h(\tau) d\tau \\
& + \left[\rho U \Gamma_0 \left(\tan^2\alpha + 1 \right) + \frac{\rho k_\alpha}{2} \frac{\partial C_{L,s}}{\partial \alpha} \left(U \tan\alpha - \frac{2\Gamma_0}{\pi c} \right) \right. \\
& \left. + \frac{\rho k_{\dot{\alpha}}}{2\tau_c} \left(U \tan\alpha - \frac{2\Gamma_0}{\pi c} \right) + \frac{m_v e \sin\alpha}{\tau_v^2} \right] \int_0^t d\tau_1 \int_0^{\tau_1} u_\alpha(\tau_2) d\tau_2 \\
& + \left[\frac{\rho k_\alpha}{2U\tau_c} \frac{\partial C_{L,s}}{\partial \alpha} \left(U \tan\alpha - \frac{2\Gamma_0}{\pi c} \right) - \frac{m_v \sin(2\alpha)}{2\tau_v^2} \right] \int_0^t d\tau_1 \int_0^{\tau_1} u_h(\tau_2) d\tau_2 \\
& - \frac{\rho}{8\pi c} \left(-\pi c^2 + 2k_{\dot{\alpha}} \right)^2 \int_0^t u_\alpha(\tau_1) d\tau_1 \int_0^{\tau_1} u_\alpha(\tau_2) d\tau_2 \\
& - \left[m_v \sin\alpha \tan\alpha + \frac{\rho k_\alpha}{4\pi c U} \frac{\partial C_{L,s}}{\partial \alpha} \left(-\pi c^2 + 2k_{\dot{\alpha}} \right) \right] \int_0^t u_\alpha(\tau_1) d\tau_1 \int_0^{\tau_1} u_h(\tau_2) d\tau_2 \\
& - \left[m_v \sin\alpha \tan\alpha + \frac{\rho k_\alpha}{4\pi c U} \frac{\partial C_{L,s}}{\partial \alpha} \left(-\pi c^2 + 2k_{\dot{\alpha}} \right) \right] \int_0^t u_h(\tau_1) d\tau_1 \int_0^{\tau_1} u_\alpha(\tau_2) d\tau_2 \\
& + \left[\frac{\rho k_\alpha}{2U^2} \frac{\partial^2 C_{L,s}}{\partial \alpha^2} \left(U \tan\alpha - \frac{2\Gamma_0}{\pi c} \right) - \frac{\rho k_\alpha^2}{2\pi c U^2} \left(\frac{\partial C_{L,s}}{\partial \alpha} \right)^2 \right] \int_0^t u_h(\tau_1) d\tau_1 \int_0^{\tau_1} u_h(\tau_2) d\tau_2
\end{aligned}
\tag{4.19}
$$

Assuming the inputs of the system, the pitching and plunging motions, to be sinusoidal with a phase shift between them like in equation 4.10, the Fliess expansion becomes

$$
\begin{aligned}
D(t) \approx{} & \rho U \Gamma_0 \tan\alpha - \frac{\rho \Gamma_0^2}{\pi c} \\
& + \left[\frac{\rho U k_{\dot\alpha} \tan\alpha}{2} + m_v \sin\alpha \left(U - \frac{e}{\tau_v}\right) - \frac{\rho \Gamma_0}{2\pi c}\left(2k_{\dot\alpha} - \pi c^2\right)\right] U_\alpha \sin(\omega t) \\
& + \left[\frac{m_v \sin(2\alpha)}{2\tau_v} + \frac{\rho k_\alpha}{2U}\frac{\partial C_{L,s}}{\partial \alpha}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right)\right] U_h \left[\sin(\omega t + \varphi) - \sin\varphi\right] \\
& + \left[\rho U \Gamma_0 \left(\tan^2\alpha + 1\right) + \frac{\rho k_\alpha}{2}\frac{\partial C_{L,s}}{\partial \alpha}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right)\right. \\
& \left. + \frac{\rho k_{\dot\alpha}}{2\tau_c}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right) + \frac{m_v e \sin\alpha}{\tau_v^2}\right] \frac{U_\alpha}{\omega}\left[1 - \cos(\omega t)\right] \\
& + \left[\frac{\rho k_\alpha}{2U\tau_c}\frac{\partial C_{L,s}}{\partial \alpha}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right) - \frac{m_v \sin(2\alpha)}{2\tau_v^2}\right] \frac{U_h}{\omega}\left[\cos\varphi - \omega t \sin\varphi - \cos(\omega t + \varphi)\right] \\
& - \frac{\rho}{8\pi c}\left(-\pi c^2 + 2k_{\dot\alpha}\right)^2 \frac{U_\alpha^2}{2}\sin^2(\omega t) - \left[m_v \sin\alpha \tan\alpha\right. \\
& \left. + \frac{\rho k_\alpha}{4\pi c U}\frac{\partial C_{L,s}}{\partial \alpha}\left(-\pi c^2 + 2k_{\dot\alpha}\right)\right] \frac{U_\alpha U_h}{4}\left[2\sin\varphi\left(\omega t - 2\sin(\omega t)\right) - \cos(2\omega t + \varphi) + \cos\varphi\right] \\
& - \left[m_v \sin\alpha\tan\alpha + \frac{\rho k_\alpha}{4\pi c U}\frac{\partial C_{L,s}}{\partial \alpha}\left(-\pi c^2 + 2k_{\dot\alpha}\right)\right] \frac{U_\alpha U_h}{4}\left[\cos\varphi - 2\omega t \sin\varphi - \cos(2\omega t + \varphi)\right] \\
& + \left[\frac{\rho k_\alpha}{2U^2}\frac{\partial^2 C_{L,s}}{\partial \alpha^2}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right) - \frac{\rho k_\alpha^2}{2\pi c U^2}\left(\frac{\partial C_{L,s}}{\partial \alpha}\right)^2\right] \frac{U_h^2}{2}\left[\sin\varphi - \sin(\omega t + \varphi)\right]^2
\end{aligned}
$$

However, as it was done for the lift, since the objective of this study is to see the effect of the unsteady motion on the airfoil, we take only the increment with respect to the steady value. And to account for the overall effect, the average of the increment over one period of time is calculated

$$
\begin{aligned}
\Delta\bar{D} \approx &-\left[\frac{m_v \sin(2\alpha)}{2\tau_v} + \frac{\rho k_\alpha}{2U}\frac{\partial C_{L,s}}{\partial\alpha}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right)\right] U_h \sin\varphi \\
&+ \left[\rho U \Gamma_0 \left(\tan^2\alpha + 1\right) + \frac{\rho k_\alpha}{2}\frac{\partial C_{L,s}}{\partial\alpha}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right)\right. \\
&\left. + \frac{\rho k_{\dot\alpha}}{2\tau_c}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right) + \frac{m_v e \sin\alpha}{\tau_v^2}\right]\frac{U_\alpha}{\omega} \\
&+ \left[\frac{\rho k_\alpha}{2U\tau_c}\frac{\partial C_{L,s}}{\partial\alpha}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right) - \frac{m_v\sin(2\alpha)}{2\tau_v^2}\right]\frac{U_h}{\omega}\left(\cos\varphi - \pi\sin\varphi\right) \\
&- \frac{\rho}{8\pi c}\left(-\pi c^2 + 2k_{\dot\alpha}\right)^2\frac{U_\alpha^2}{4} \\
&- \left[m_v\sin\alpha\tan\alpha + \frac{\rho k_\alpha}{4\pi c U}\frac{\partial C_{L,s}}{\partial\alpha}\left(-\pi c^2 + 2k_{\dot\alpha}\right)\right]\frac{U_\alpha U_h}{2}\cos\varphi \\
&+ \left[\frac{\rho k_\alpha}{2U^2}\frac{\partial^2 C_{L,s}}{\partial\alpha^2}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right) - \frac{\rho k_\alpha^2}{2\pi c U^2}\left(\frac{\partial C_{L,s}}{\partial\alpha}\right)^2\right]\frac{U_h^2}{2}\left[1 - \frac{1}{2}\cos(2\varphi)\right]
\end{aligned} \tag{4.20}
$$

Since the inputs that are considered are of high-frequency, some of the terms that appear in the equation can be neglected in favor of those that have a more important contribution, leading to

$$
\begin{aligned}
\Delta\bar{D} \sim &-\left[\frac{m_v \sin(2\alpha)}{2\tau_v} + \frac{\rho k_\alpha}{2U}\frac{\partial C_{L,s}}{\partial\alpha}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right)\right] U_h \sin\varphi \\
&- \frac{\rho}{8\pi c}\left(-\pi c^2 + 2k_{\dot\alpha}\right)^2\frac{U_\alpha^2}{4} \\
&- \left[m_v\sin\alpha\tan\alpha + \frac{\rho k_\alpha}{4\pi c U}\frac{\partial C_{L,s}}{\partial\alpha}\left(-\pi c^2 + 2k_{\dot\alpha}\right)\right]\frac{U_\alpha U_h}{2}\cos\varphi \\
&+ \left[\frac{\rho k_\alpha}{2U^2}\frac{\partial^2 C_{L,s}}{\partial\alpha^2}\left(U\tan\alpha - \frac{2\Gamma_0}{\pi c}\right) - \frac{\rho k_\alpha^2}{2\pi c U^2}\left(\frac{\partial C_{L,s}}{\partial\alpha}\right)^2\right]\frac{U_h^2}{2}\left[1 - \frac{1}{2}\cos(2\varphi)\right]
\end{aligned} \tag{4.21}
$$

According to this approximation, when the pitching and plunging motions are in phase there are only second order effects. The contribution of the plunging motion appears through the suction force and, as it can be seen, is always negative. Therefore, it generates a propulsive force that counteracts the effect of the drag. However, its magnitude depends on the difference between the terms that are inside the parenthesis. According to the model used, $k_{\dot{\alpha}} = \pi c^2 \left(\frac{3}{4} - \frac{e}{c}\right)$, where e is the position of the pitching axis. Therefore, the plunging contribution reduces to $-\frac{\rho \pi c^3}{2} \left(\frac{1}{4} - \frac{e}{c}\right)^2$, meaning that the propulsive force increases as the pitching axis moves away from the quarter chord point.

A similar thing occurs with the interaction between the pitching and plunging motions. It consists of two terms, the first one is given by the pressure drag, and the second one by the suction force. It seems that the overall contribution is negative, inducing a force in the forward direction. The pressure term, proportional to the added mass, appears due to the non-circulatory lift and is always positive for positive angles of attack, contributing therefore to the propulsive force. Nonetheless, the second term depends on the first derivative of the steady lift coefficient with respect to the angle of attack but also on the difference between the terms that are in the parenthesis. As it has been stated in the paragraph, it means that this suction term will be positive or negative depending on the first derivative of the static lift curve but also on the position of the pitching axis. Therefore, the total contribution of the interaction between motions is inconclusive, and it could be positive or negative, generating drag or thrust.

Finally, the effect of the plunging motion alone is given completely by the suction force and depends on the first and second derivatives of the static lift coefficient with respect to the angle of attack. The second term is always negative, contributing directly to the propulsive force. However the sign of the first term depends on the value of the variables inside the brackets, which, substituting the value of the quasi-steady circulation at the equilibrium point, become $U\left(\tan\alpha - \frac{C_{L,s}(\alpha)}{\pi}\right)$. In other words, the plunging contribution may be drag or thrust depending on the angle of attack of the airfoil.

Regarding the effect of the phase shift between the pitching and plunging motions, the major difference is the addition of a linear term. It is negligible for small phase differences but it increases as the absolute value of the phase difference increases. It reaches its maximum absolute value at $\varphi = 90°$. However, when the phase difference is positive it induces drag, but if the phase difference is negative it generates thrust. About its value, the first term of this contribution is always positive for positive angles of attack, so it contributes to drag reduction. But the second term depends strongly on the angle of attack, so its sign is uncertain. Therefore, it is unclear if the effect of this linear term is to increase or to reduce the drag.

4.4 Circulation

According to the Kutta-Joukowsky theorem, in a steady flow, the lift force is defined in the following way

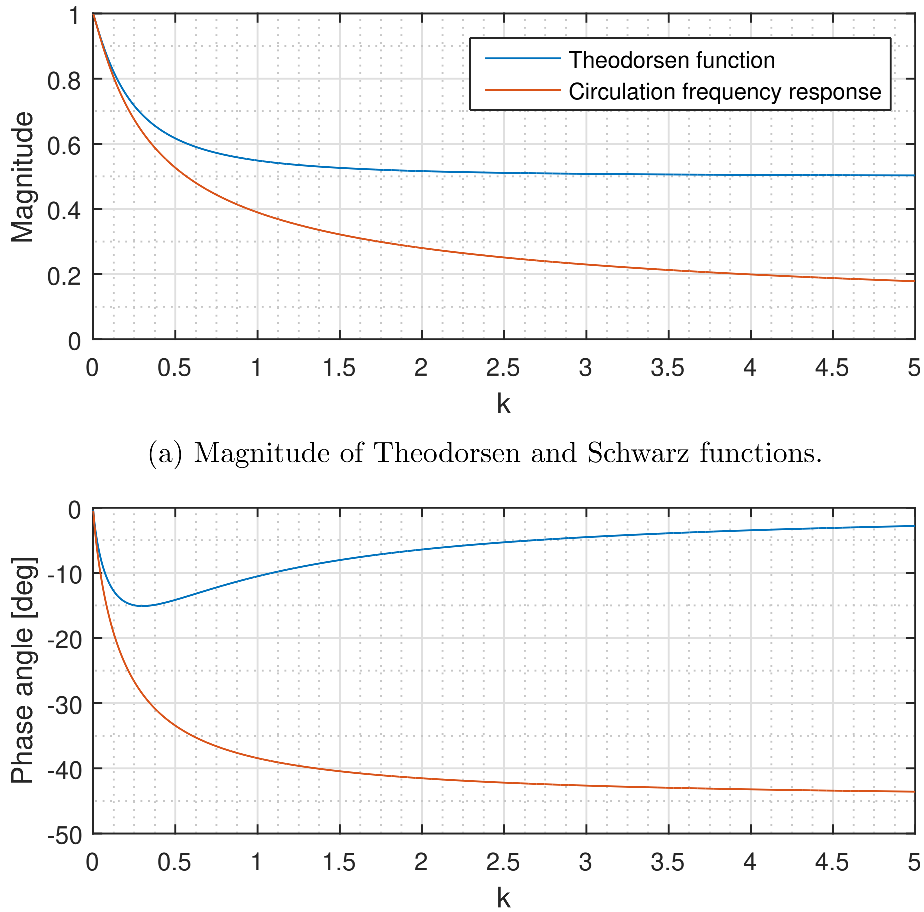
$$L = \rho U \Gamma$$

In other words, in a steady flow the circulation leads to the development of lift. But it may not be that straightforward in the case of unsteady aerodynamics. The frequency response of the circulation is not the Theodorsen function $C(k)$. It is given by Schwarz [40]

$$\frac{\Gamma}{\Gamma_{QS}}(k) = \frac{-2e^{-jk}}{jk\pi\left(H_1^{(2)}(k) + jH_0^{(2)}(k)\right)}$$

where $H_n^{(m)}$ is the Hankel function of m^{th} kind of order n.

Figure 4.3 draws a comparison between the Schwarz and Theodorsen functions. As it was pointed out in section 2.1, the high frequency gain of the Theodorsen function is equal to $\frac{1}{2}$, meaning that when the airfoil changes its angle of attack, it instantaneously obtains half of its quasi-steady circulation. This characteristic is not realistic for a dynamical system, in which there is always some lag between the input and the output. However, the Schwarz function tends to zero as the reduced frequency increases. That is, when the airfoil changes its motion, there is no instantaneous change in the circulation, it builds up over time, as it would be expected in a real aerodynamic problem.



(a) Magnitude of Theodorsen and Schwarz functions.

(b) Phase of Theodorsen and Schwarz functions.

Figure 4.3: Comparison between the transfer function of the circulatory lift (Theodorsen) and the transfer function of the circulation (Schwarz).

On the other hand, since the objective of this study is to develop a state space system, it is necessary to obtain the relation between the circulation and its quasi-steady value in the time domain. Luckily, Sears [41] observed that the indicial response for the circulation is given by the Küssner function [42]

$$\psi(\tau) = 1 - \int_0^\infty \frac{\left(I_0(x) + I_1(x)\right) \mathrm{e}^{-x(\tau-1)} dx/x^2}{\left(K_0(x) - K_1(x)\right)^2 + \pi^2 \left(I_0(x) + I_1(x)\right)^2}$$

where I_n are modified Bessel functions of first kind and K_n are modified Bessel functions of second kind.

This expression can be seen as the equivalent to the Wagner function but for the circulation. For this reason, Sears [43], inspired by Jones [19], approximated the function to an exponential function, in order to have an analytical expression

$$\psi\left(s\right) = 1 - A_{\psi 1}\mathrm{e}^{-b_{\psi 1}s} - A_{\psi 2}\mathrm{e}^{-b_{\psi 2}s}$$

with $A_{\psi 1} = A_{\psi 2} = 0.5$, $b_{\psi 1} = 0.13$ and $b_{\psi 2} = 1$; and s being the non-dimensional time, as it is defined in section 2.2.

Since the approximation is very similar to those of Jones [19] and Jones [20] for the circulatory lift, the circulation can be treated in a similar way. Therefore, the Duhamel principle can be applied to Γ as it was done to L_C in section 2.3

$$\Gamma(t) = -\int_0^t \Gamma_0(\tau)\frac{d\psi\left(t-\tau\right)}{d\tau}d\tau$$

From here, following the same development of Taha et al. [21] for the circulatory lift, Γ can be obtained by solving the set of differential equations

$$\dot{x}_i(t) = \frac{2b_{\psi i}U(t)}{c}\left(-x_i(t) + A_{\psi i}\Gamma_0(t)\right), \qquad i = 1, 2 \tag{4.22}$$

where $U(t)$ is the free stream velocity, c the airfoil chord, and Γ_0 the quasi-steady circulation; and the output, the circulation, is given by

$$\Gamma(t) = x_1(t) + x_2(t)$$

That is, Γ follows a dynamical behavior very similar to that of L_C obtained in section 2.3. And, as it occurred for the circulatory lift, the build up of the circulation follows a first order dynamical equation with only two states. The same derivation of this model can be found in [44].

Since the expression is equivalent to that of the circulatory lift, it can also be approximated to just one state. As it is stated in section 2.3, the addition or deletion of a state does not change the behavior of the system, therefore, approximating the system to just one state allows to simplify the calculations without compromising the validity of the results. Therefore, the equation that has to be added to the state space system is the following

$$\dot{x}_\Gamma = -\frac{1}{\tau_\Gamma} x_\Gamma + \frac{1}{\tau_\Gamma} \Gamma_0 \left(\alpha, \dot{\alpha}, \dot{h} \right) \tag{4.23}$$

where the output is given directly by the new state $\Gamma = x_\Gamma$.

4.4.1 Lie derivatives

In order to compute the effect of the unsteady motion on the development of the circulation, the rate of change of Γ due to different motions is computed. That is, the Lie derivatives of the circulation along the directions given by the symmetric products are calculated and evaluated at the equilibrium point of the averaged dynamics

$$\mathcal{L}_{\langle \boldsymbol{g_\alpha} : \boldsymbol{g_\alpha} \rangle} \Gamma = 0 \tag{4.24}$$

$$\mathcal{L}_{\langle \boldsymbol{g_\alpha} : \boldsymbol{g_h} \rangle} \Gamma = 0 \tag{4.25}$$

$$\mathcal{L}_{\langle \boldsymbol{g_h} : \boldsymbol{g_h} \rangle} \Gamma = -\frac{k_\alpha}{\tau_\Gamma U^2} \frac{\partial^2 C_{L,s}}{\partial \alpha^2} \tag{4.26}$$

According to the results of the Lie derivatives, the pitching motion has no effect on the circulation of the airfoil. At the same time, there seems to be no interaction between both motions, pitching and plunging, regarding the development of Γ.

On the other hand, the pitching motion seems to have a net effect on the circulation that is proportional to the second derivative of the static lift coefficient with respect to the angle of attack. As it is discussed in section 4.2.1, in a conventional airfoil this term is zero in the linear regime, so there will only be this enhancement on the post-stall region. Nonetheless, on a delta wing, since the lift curve is convex for its whole range of angles of attack, this contribution is always positive.

4.4.2 Fliess functional expansion

Another geometric tool that is useful to analyze the unsteady effect on the airfoil circulation is the Fliess functional analysis. This input-output representation provides the means to analyze the response due to pitching and plunging in a more direct and intuitive way.

The second order Fliess functional expansion of the circulation has the following form

$$\Gamma(t) \approx \Gamma_0 + \frac{k_{\dot{\alpha}}}{\tau_\Gamma} \int_0^t d\tau_1 \int_0^{\tau_1} u_\alpha(\tau_2) d\tau_2 + \frac{k_\alpha}{\tau_\Gamma U} \frac{\partial C_{L,s}}{\partial \alpha} \int_0^t d\tau_1 \int_0^{\tau_1} u_h(\tau_2) d\tau_2 \tag{4.27}$$

This is the simplest expansion of all the output variables that are being studied in this project. The first order terms are canceled. In other words, there is only a change in the value of the circulation if, at least, second order effects are considered.

Assuming the pitching and plunging motions to be sinusoidal in the form of equation 4.10, the series become

$$\Gamma(t) \approx \Gamma_0 + \frac{k_{\dot{\alpha}}}{\tau_\Gamma}\frac{U_\alpha}{\omega}\left[1-\cos\left(\omega t\right)\right] + \frac{k_\alpha}{\tau_\Gamma U}\frac{\partial C_{L,s}}{\partial \alpha}\frac{U_h}{\omega}\left[\cos\varphi - \omega t\sin\varphi - \cos\left(\omega t + \varphi\right)\right]$$

Since periodic fluctuations of the circulation are not of our interest, but the total increase or decrease of its effective value, the expression is averaged over one period of time. Aside from that, the series can be written in the form of the steady value plus its variation due to the unsteady motion $\Gamma = \Gamma_0 + \Delta\Gamma$. Taking only the average of the variation

$$\Delta\bar{\Gamma} \approx \frac{k_{\dot{\alpha}}}{\tau_\Gamma}\frac{U_\alpha}{\omega} + \frac{k_\alpha}{\tau_\Gamma U}\frac{\partial C_{L,s}}{\partial \alpha}\frac{U_h}{\omega}\left(\cos\varphi - \pi\sin\varphi\right) \tag{4.28}$$

In this case there is no term that appears due to the interaction of both motions. However, the pitching motion leads to an increase on the overall value of the circulation. It is a constant, it does not depend on parameters such as the angle of attack, only on the frequency of the motion.

On the other hand, the increment due to the plunging motion has a more complicated expression. It is proportional to the first derivative of the static lift coefficient with respect to the angle of attack. When there is no phase shift between the motions, for delta wings the derivative is always positive, meaning that the circulation increases. Still, in a conventional airfoil, this first derivative is positive or negative depending on the angle of attack. Consequently, it cannot be easily determined if this term increases or decreases the overall circulation of the wing.

Regarding the phase shift, it only affects the plunging motion. It can determine whereas the contribution of the plunging is positive or negative, depending on the sign of $\cos\varphi - \pi\sin\varphi$. For angles ranging from $-180°$ to $17.66°$ this term is positive, and from $17.66°$ to $180°$ it is negative. Combining it with the sign of the first derivative of the static lift coefficient with respect to the angle of attack, it will be possible to determine if there is an increase or a decrease of the circulation due to plunging.

Nonetheless, the inputs that are studied in this project are high-frequency oscillations. Therefore, the contributions of pitching and plunging as they appear in equation 4.28 may be very small, so that they can be neglected in most of the cases.

4.5 Point of separation

One important aerodynamic characteristic that most studies do not take into account is the point at which the flow separates from the airfoil. In ideal conditions it is assumed that the flow smoothly leaves the airfoil at the trailing edge, and is attached to its whole surface. Unfortunately, this is not always the case for a real viscous flow.

As the angle of attack of the airfoil increases, the flow tends to separate from the top surface, creating a wake behind it [45]. In this separated region the flow can recirculate, so that part of it moves in the direction opposite to the free stream velocity, as it occurs in figure 4.4. This flow is called the reverse flow. When this flow separation phenomenon occurs, there is an important decrease in lift, and a high increase in the drag force.

For obvious reasons, it is in the interest of the aerodynamicists to avoid flow separation and, in case that it occurs (it will certainly do), to keep the point of separation of the flow as close to the trailing edge as possible.

In steady aerodynamics one of the most common approaches to define the location of the separation point is that of Thwaites [46]. Nonetheless, his method is not suitable for this study because it requires previous knowledge of the velocity profile around the airfoil and,

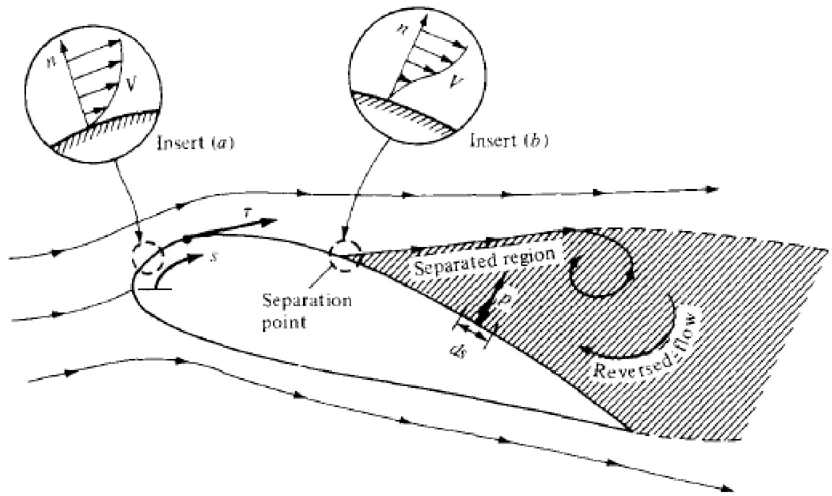


Figure 4.4: Flow separation at the airfoil [45].

aside of that, its resolution implies the integration of several complicated expressions. Since it is of our interest to find a simpler expression that can be easily analyzed in the state space model, it is preferable to apply Kirchhoff model. According to this theory, the lift coefficient varies with the location of the separation point in the following form

$$C_{L,s}(\alpha) = 2\pi\alpha \left(\frac{1 + \sqrt{x_{s,s}}}{2} \right)^2$$

where $x_{s,s}$ is the location of the point of separation normalized by the airfoil chord, and $2\pi\alpha$ is the theoretical expression of the lift curve when there is no separation.

Therefore, if the static lift coefficient is known, the separation point is given by

$$x_{s,s} = \left(\sqrt{\frac{2C_{L,s}(\alpha)}{\pi\alpha}} - 1 \right)^2 \tag{4.29}$$

Unfortunately, this expression is only valid for steady flows. For unsteady motions Goman and Khrabrov [47] derived a state space solution for a pitching airfoil. According to them, the unsteady effects can be divided in two different types. The first one accounts for the quasi-steady effects, such as the lag of the circulation or the convection of the boundary layer. The delay due to these conditions is proportional to the rate of variation of the angle of attack $\dot{\alpha}$. Therefore, the quasi-steady variation of the point of separation $x_{s,0}\alpha$ is simply modeled as an argument shift $x_{s,0}\left(\alpha - \tau_2\dot{\alpha}\right)$, where τ_2 is the time delay constant.

On the other hand, there are also transient aerodynamic effects that cannot be defined as quasi-steady. In this group are included the disturbances that occur without a variation on the angle of attack. They lead to a build up from the state previous to the perturbation to the steady state after the perturbation. This process is modeled as a first order dynamical system with the relaxation constant τ_1.

The motion of the point of separation x_s for an unsteady flow is approximated by the following equation

$$\tau_1 \frac{dx_s}{dt} + x_s = x_{s,0}\left(\alpha - \tau_2\dot{\alpha}\right) \tag{4.30}$$

In order to adapt the following expression to account also for the plunging motion it is necessary to change the effective angle of attack to $\alpha + \arctan\frac{\dot{h}}{U}$. Applying this modification, combining the expressions 4.29 and 4.30 and rewriting the equation in the form of the state space system

$$\dot{x}_s = -\frac{1}{\tau_1}x_s + \frac{1}{\tau_1}\left(\sqrt{\frac{2C_{L,s}\left(\alpha + \arctan\frac{\dot{h}}{U} - \tau_2\dot{\alpha}\right)}{\pi\left(\alpha + \arctan\frac{\dot{h}}{U} - \tau_2\dot{\alpha}\right)}} - 1\right)^2 \tag{4.31}$$

Therefore, the point of separation x_s is defined as a new state of the system, modeled by this simple first order equation. The parameters τ_1 and τ_2 are empirical constants that strongly depend on the case of study.

4.5.1 Lie derivatives

In order to find the overall effects of the pitching and plunging motions on the position of the separation point, its rate of change along the dynamics of these motions is computed. These results are given by the following Lie derivatives, which are evaluated at the equilibrium point of the averaged dynamics

$$\mathcal{L}_{\langle \boldsymbol{g_\alpha}:\boldsymbol{g_\alpha}\rangle} x_s = -\frac{\tau_2^2}{\tau_1}\sqrt{\frac{2}{\pi\alpha C_{L,s}}}\left[\left(\sqrt{\frac{2C_{L,s}}{\pi\alpha}}-1\right)\left(\frac{\partial^2 C_{L,s}}{\partial\alpha^2}-\frac{2}{\alpha}\frac{\partial C_{L,s}}{\partial\alpha}+\frac{2C_{L,s}}{\alpha^2}\right)\right.$$
$$\left. + \frac{1}{2C_{L,s}}\left(\frac{\partial C_{L,s}}{\partial\alpha}-\frac{C_{L,s}}{\alpha}\right)^2\right] \quad (4.32)$$

$$\mathcal{L}_{\langle \boldsymbol{g_\alpha}:\boldsymbol{g_h}\rangle} x_s = \frac{\tau_2}{\tau_1 U}\sqrt{\frac{2}{\pi\alpha C_{L,s}}}\left[\left(\sqrt{\frac{2C_{L,s}}{\pi\alpha}}-1\right)\left(\frac{\partial^2 C_{L,s}}{\partial\alpha^2}-\frac{2}{\alpha}\frac{\partial C_{L,s}}{\partial\alpha}+\frac{2C_{L,s}}{\alpha^2}\right)\right.$$
$$\left. + \frac{1}{2C_{L,s}}\left(\frac{\partial C_{L,s}}{\partial\alpha}-\frac{C_{L,s}}{\alpha}\right)^2\right] \quad (4.33)$$

$$\mathcal{L}_{\langle \boldsymbol{g_h}:\boldsymbol{g_h}\rangle} x_s = -\frac{1}{\tau_1 U^2}\sqrt{\frac{2}{\pi\alpha C_{L,s}}}\left[\left(\sqrt{\frac{2C_{L,s}}{\pi\alpha}}-1\right)\left(\frac{\partial^2 C_{L,s}}{\partial\alpha^2}-\frac{2}{\alpha}\frac{\partial C_{L,s}}{\partial\alpha}+\frac{2C_{L,s}}{\alpha^2}\right)\right.$$
$$\left. + \frac{1}{2C_{L,s}}\left(\frac{\partial C_{L,s}}{\partial\alpha}-\frac{C_{L,s}}{\alpha}\right)^2\right] \quad (4.34)$$

These expressions are more complex than the ones obtained for the other outputs. However, it can be stated that all the Lie derivatives are zero in the linear regime of a conventional airfoil, assuming that in this range $C_{L,s} = 2\pi\alpha$. Therefore, there is no variation in the position of the separation point due to a pitching or plunging motion of the wing in the linear regime. Nonetheless, in the nonlinear region the expressions become more interesting.

There seems to be a variation in the location of the separation point due to pitching, plunging and their interaction. In all cases the magnitude of the displacement appears to be similar, and strongly dependent on the steady lift coefficient and its first and second derivatives with respect to the angle of attack.

Since the Lie derivatives are too complex to be fully understood, they are simplified in the following lines

$$\mathcal{L}_{\langle \boldsymbol{g_\alpha} : \boldsymbol{g_\alpha} \rangle} x_s = -\frac{\tau_2^2}{\tau_1} \frac{\partial^2 x_{s,s}}{\partial \alpha^2} \tag{4.35}$$

$$\mathcal{L}_{\langle \boldsymbol{g_\alpha} : \boldsymbol{g_h} \rangle} x_s = \frac{\tau_2}{\tau_1 U} \frac{\partial^2 x_{s,s}}{\partial \alpha^2} \tag{4.36}$$

$$\mathcal{L}_{\langle \boldsymbol{g_h} : \boldsymbol{g_h} \rangle} x_s = -\frac{1}{\tau_1 U^2} \frac{\partial^2 x_{s,s}}{\partial \alpha^2} \tag{4.37}$$

All the derivatives are proportional to the second derivative of the location of the separation point in steady conditions. As it can be seen in figure 4.5, in conventional airfoils this derivative starts with a value of zero and becomes negative as the angle of attack increases. However, after a certain angle (approximately 11° in a NACA 0012 airfoil) it becomes positive. Therefore, the pitching and plunging motions of the airfoil do have an effect on it. However, to determine if it displaces the separation point forwards or backwards it is necessary to look at the Fliess functional expansion.

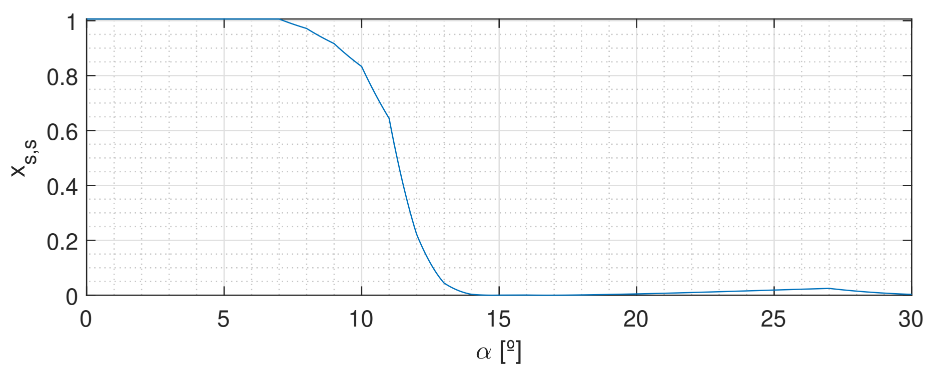


Figure 4.5: Steady location of the trailing-edge separation point in a NACA0012 airfoil.

4.5.2 Fliess functional expansion

In order to determine if the effect of the unsteady motion is positive or negative on the location of the separation point, its Fliess functional expansion is computed. Since the unsteady outcomes are nonlinear, it is necessary to compute at least a second order approximation, as it is done in the following lines.

$$\begin{aligned} x_s \approx x_{s,0}(\alpha) &- \frac{\tau_2}{\tau_1}\sqrt{\frac{2}{\pi\alpha C_{L,s}}}\left(\sqrt{\frac{2C_{L,s}}{\pi\alpha}}-1\right)\left(\frac{\partial C_{L,s}}{\partial\alpha}-\frac{C_{L,s}}{\alpha}\right)\int_0^t d\tau_1\int_0^{\tau_1} u_\alpha(\tau_2)d\tau_2 \\ &+ \frac{1}{\tau_1 U}\sqrt{\frac{2}{\pi\alpha C_{L,s}}}\left(\sqrt{\frac{2C_{L,s}}{\pi\alpha}}-1\right)\left(\frac{\partial C_{L,s}}{\partial\alpha}-\frac{C_{L,s}}{\alpha}\right)\int_0^t d\tau_1\int_0^{\tau_1} u_h(\tau_2)d\tau_2 \end{aligned} \quad (4.38)$$

As it occurred in the case of the circulation the first order terms are canceled. Consequently, only the second order terms have an effect on the location of the separation point. Assuming the pitching and plunging motions to be sinusoidal in the form of equation 4.10, the Fliess expansion becomes

$$\begin{aligned} x_s \approx x_{s,0}(\alpha) &- \frac{\tau_2}{\tau_1}\sqrt{\frac{2}{\pi\alpha C_{L,s}}}\left(\sqrt{\frac{2C_{L,s}}{\pi\alpha}}-1\right)\left(\frac{\partial C_{L,s}}{\partial\alpha}-\frac{C_{L,s}}{\alpha}\right)\frac{U_\alpha}{\omega}\left[1-\cos\left(\omega t\right)\right] \\ &+ \frac{1}{\tau_1 U}\sqrt{\frac{2}{\pi\alpha C_{L,s}}}\left(\sqrt{\frac{2C_{L,s}}{\pi\alpha}}-1\right)\left(\frac{\partial C_{L,s}}{\partial\alpha}-\frac{C_{L,s}}{\alpha}\right)\frac{U_h}{\omega}\left[\cos\varphi-\omega t\sin\varphi-\cos\left(\omega t+\varphi\right)\right] \end{aligned}$$

However, since the interest is not in the real location of the separation point but on its variation due to the unsteady motion, it is useful to take only the increment with respect to the steady value. Aside from that, in order to evaluate the overall effect of the pitching and the plunging and not the instantaneous variations, the expression is averaged over one period of time.

$$\Delta\bar{x}_s \approx -\frac{\tau_2}{\tau_1}\sqrt{\frac{2}{\pi\alpha C_{L,s}}}\left(\sqrt{\frac{2C_{L,s}}{\pi\alpha}}-1\right)\left(\frac{\partial C_{L,s}}{\partial\alpha}-\frac{C_{L,s}}{\alpha}\right)\frac{U_\alpha}{\omega} + \frac{1}{\tau_1 U}\sqrt{\frac{2}{\pi\alpha C_{L,s}}}\left(\sqrt{\frac{2C_{L,s}}{\pi\alpha}}-1\right)\left(\frac{\partial C_{L,s}}{\partial\alpha}-\frac{C_{L,s}}{\alpha}\right)\frac{U_h}{\omega}\left(\cos\varphi-\pi\sin\varphi\right) \quad (4.39)$$

As it can be seen, the effect of the pitching and the plunging motions in the location of the separation point is the opposite. Whereas one moves it forwards the other one shifts it backwards. However, the effect depends on the sign of the terms in brackets, as it can be seen in figure 4.6 for a NACA 0012 airfoil. For a conventional airfoil this product is zero for low angles of attack, but it becomes rapidly negative past 5°. However, it is positive for the range that goes approximately from 17° to 26°, which is in the post-stall regime, the region in which this study is centered.

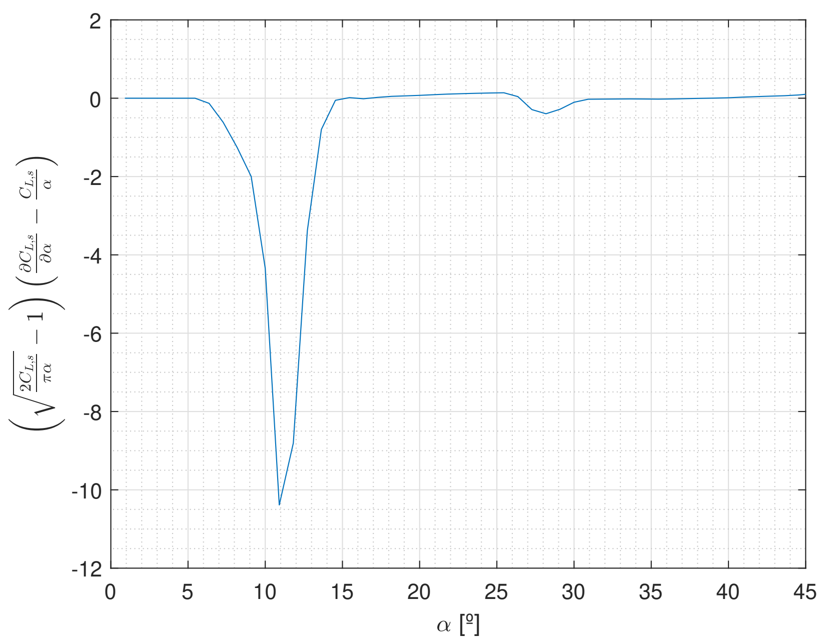


Figure 4.6: Terms in brackets for a NACA0012 airfoil at $Re = 50,000$.

Therefore, if the angle of attack is between $17-26^{\circ}$ the pitching motion moves the separation point forwards, but the plunging motion moves it backwards. Nonetheless, the behavior is the opposite outside the mentioned range.

As for the effect of the phase shift between the inputs, if the phase difference is bigger than $\varphi = \arctan \frac{1}{\pi} = 17.66^{\circ}$ the effect of the plunging motion is reversed, so that both movements have the same effect on the wing. In other words, for a NACA 0012 airfoil, if the angle of attack is in the range $17-26^{\circ}$, the unsteady motion will move the separation point forwards, increasing the surface with separated flow. Nonetheless, outside this range, both pitching and plunging will shift the separation point backwards, making for more attached flow on the airfoil.

Chapter 5

Conclusions and future work

5.1 Conclusions

It is possible to develop a reduced order model (ROM) for the unsteady flow around a two-dimensional pitching and plunging airfoil that captures most of the physics of the flow without entirely reconstructing it. The nonlinearities of the flow at large angles of attack and high frequencies is captured in great measure by this simple and computationally inexpensive state space model.

One of the most important conclusions that can be extracted from this ROM is the lift enhancement that appears when the airfoil is subject to certain motions. Specifically, the plunging motion seems to induce an extra lift force that depends on the second derivative of the lift coefficient with respect to the angle of attack. Therefore, in a conventional airfoil this enhancement would only appear in the post stall regime, but in a delta wing it would be present in the whole range of operation.

Another remarkable result is that of the drag force. It appears that both the pitching and the plunging motions generate a propulsive force that counteracts the effects of the aerodynamic drag. The magnitude of this generated force depends, again, on the first and second derivatives of the lift coefficient with respect to the angle of attack. Depending on the parameters of the airfoil and the conditions of flight this counteracting force can overcome the drag resulting in a thrust force. In other words, the it may be possible to generate thrust by simply inducing a pitching and/or plunging motion on an airfoil.

Regarding the circulation, pitching is seen to increase its value, whereas plunging may have a positive or negative contribution depending on the sign of the first derivative of the lift with respect to the angle of attack.

Finally, the effect of pitching and plunging on the point of separation appears to be opposite, and depends on the lift coefficient and its first derivative with respect to the angle of attack. In fact, if the value of the angle of attack is between $17 - 26°$ the pitching motion moves the separation point forwards, but the plunging motion moves it backwards. However, the effect is the opposite outside of the mentioned range.

5.2 Future work

First of all, the most important work to be done is to validate the obtained results. In order to do that, several simulations have to be done. They should include pure-pitching, pure-plunging and combined motion at different angles of attack and Reynolds numbers. It would also be recommendable to study different reduced frequencies and amplitudes of movement. On the other hand, if the simulations lead to positive results, the last validation that should be done would be to make some experiments.

Secondly, other inputs could be considered. For example, surging and flapping motions could be added to the system to see if they have a considerable effect on the lift, thrust, circulation or location of the point of separation. Furthermore, other more complicated waveforms could be considered as inputs: triangle wave, pulse train, etc. Also, it would be interesting to study the effects on the system when the motions have different frequencies.

Finally, some other interesting outputs may be considered, e.g. the pitching moment. Even though this addition of variables may complicate the state space system, it could lead to a better comprehension of the flow, which could be essential in understanding the processes that occur when an airfoil or a wing moves in an unsteady motion.

Bibliography

[1] T. Theodorsen, "General theory of aerodynamic instability and the mechanism of flutter," Tech. Rep. NACA-TR-496, National Advisory Committe for Aeronautics, Hampton, 1935.

[2] H. Wagner, "Über die Entstehung des dynamischen Auftriebes von Tragflügeln," *Mathematica*, vol. 5, pp. 17–35, 1925.

[3] A. G. Rainey, "Measurement of aerodynamic forces for various mean angles of attack on an airfoil oscillating in pitch and on two finite-span wings oscillating in bending with emphasis on damping in the stall," Tech. Rep. NACA-TR-1305, National Advisory Committee for Aeronautics, Langley Field, VA, 1957.

[4] N. Vandenberghe, J. Zhang, and S. Childress, "Symmetry breaking leads to forward flapping flight," *Journal of Fluid Mechanics*, vol. 506, no. 506, pp. 147–155, 2004.

[5] D. Rival and C. Tropea, "Characteristics of pitching and plunging airfoils under dynamic-stall conditions," *Journal of Aircraft*, vol. 47, no. 1, pp. 80–86, 2010.

[6] D. E. Rival, J. Kriegseis, P. Schaub, A. Widmann, and C. Tropea, "Characteristic length scales for vortex detachment on plunging profiles with varying leading-edge geometry," *Experiments in Fluids*, vol. 55, no. 1, pp. 1–8, 2014.

[7] S. Heathcote and I. Gursul, "Jet switching phenomenon for a periodically plunging airfoil," *Physics of Fluids*, vol. 19, no. 2, 2007.

[8] H. E. Taha, "Geometric control theoretic formulation and analysis of unsteady fluid flows," 2018.

[9] R. W. Brockett, "System theory on group manifolds and coset spaces," *SIAM Journal on Control and Optimization*, vol. 10, no. 2, pp. 265–284, 1972.

[10] R. W. Brockett, "Nonlinear Systems and Differential Geometry," *Proceedings of the IEEE*, vol. 64, no. 1, pp. 61–72, 1976.

[11] R. W. Brockett, "Control theory and singular Riemannian geometry," in *New directions in applied mathematics* (P. J. Hilton and G. S. Young, eds.), pp. 11–27, New York, NY: Springer New York, 1982.

[12] R. W. Brockett, "Asymptotic stability and feedback stabilization," in *Differential geometric control theory*, pp. 181–191, Birkhauser, 1983.

[13] H. J. Sussmann, "Orbits of families of vector fields and integrability of distributions," *Transactions of the American Mathematical Society*, vol. 180, pp. 171–188, 1973.

[14] H. J. Sussmann and V. Jurdjevic, "Controllability of nonlinear systems," *Journal of Differential Equations*, vol. 12, no. 1, pp. 95–116, 1972.

[15] H. J. Sussmann, "A general theorem on local controllability," *SIAM Journal on Control and Optimization*, vol. 25, no. 1, pp. 158–194, 1987.

[16] G. C. Walsh and S. Sastry, "On reorienting linked rigid bodies using internal motions," in *Proceedings on the 30th Conference on Decision and Control*, (Brighton, England), pp. 1190–1195, 1991.

[17] P. E. Crouch, "Spacecraft attitude control and stabilization: Applications of geometric control theory to rigid body models," *IEEE Transactions on Automatic Control*, vol. 29, no. 4, pp. 321–331, 1984.

[18] I. E. Garrick, "On some reciprocal relations in the theory of nonstationary flows," Tech. Rep. NACA-TR-629, National Advisory Committee for Aeronautics, Langley Field, VA, 1938.

[19] R. T. Jones, "Operational treatment of the nonuniform-lift theory in airplane dynamics," Tech. Rep. NACA-TR-667, National Advisory Committee for Aeronautics, Washington, D.C., 1938.

[20] W. P. Jones, "Aerodynamic forces on wings in non-uniform motion," Tech. Rep. 2117, British Aeronautical Research Council, 1945.

[21] H. E. Taha, M. R. Hajj, and P. S. Beran, "State-space representation of the unsteady aerodynamics of flapping flight," *Aerospace Science and Technology*, vol. 34, no. 1, pp. 1–11, 2014.

[22] F. Bullo and A. D. Lewis, *Geometric control of mechanical systems: modeling, analysis and design for symple mechanical control systems*, vol. 49. New York: Springer, 2004.

[23] S. Sastry, *Nonlinear systems: Analysis, stability and control*, vol. 10. New York: Springer International Publishing, 1999.

[24] R. M. Murray and S. S. Sastry, "Nonholonomic motion planning. Steering using sinusoids," *IEEE Transactions on Automatic Control*, vol. 38, no. 5, pp. 700–716, 1993.

[25] W. Liu, "An approximation algorithm for nonholonomic systems," *SIAM Journal on Control and Optimization*, vol. 35, no. 4, pp. 1328–1365, 1997.

[26] W. Liu, "Averaging theorems for highly oscillatory differential equations and iterated Lie brackets," *SIAM Journal on Control and Optimization*, vol. 35, no. 6, pp. 1989–2020, 1997.

[27] J. E. Marsden, "Geometric foundations of motion and control," in *National Academy Symposium*, pp. 1–14, 1994.

[28] F. Bullo, "Averaging and vibrational control of mechanical systems," *SIAM Journal on Control and Optimization*, vol. 41, no. 2, pp. 542–562, 2003.

[29] P. A. Vela, *Averaging and control of nonlinear systems.* Phd, California Institute of Technology, Pasadena, CA, 2003.

[30] A. A. Agračev and R. V. Gamkrelidze, "The exponential representation of flows and the chronological calculus," *Matematicheskii Sbornik*, vol. 149, no. 4, pp. 467–532, 1978.

[31] H. E. Taha, C. A. Woolsey, and M. R. Hajj, "Geometric control approach to longitudinal stability of flapping flight," *Journal of Guidance, Control, and Dynamics*, vol. 39, no. 2, pp. 214–226, 2016.

[32] D. Cheng, X. Hu, and T. Shen, *Analysis and design of nonlinear control systems.* Berlin Heidelberg: Springer-Verlag, 1 ed., 2010.

[33] M. Fliess, M. Lamnabhi, and F. Lamnabhi-Lagarrigue, "An algebraic approach to nonlinear functional expansions," *IEEE Transactions on Circuits and Systems*, vol. 30, no. 8, pp. 554–570, 1983.

[34] H. E. Taha and A. S. Rezaei, "Viscous extension of potential-flow unsteady aerodynamics: The lift frequency response problem," *Journal of Fluid Mechanics*, vol. 868, pp. 141–175, 2019.

[35] R. E. Sheldahl and P. C. Klimas, "Aerodynamic characteristics of seven symmetrical airfoil sections through 180-degree angle of attack for use in aerodynamic analysis of vertical axis wind turbines.," Tech. Rep. SAND80-2114, Sandia National Laboratories, Albuquerque, NM, 1981.

[36] H. K. Cheng, "Remarks on nonlinear lift and vortex separation," *Journal of the Aeronautical Sciences*, vol. 3, no. 21, pp. 212–214, 1954.

[37] I. E. Garrick, "Propulsion of a flapping and oscillating airfoil," Tech. Rep. NACA-TR-567, National Advisory Committee for Aeronautics, Hampton, 1936.

[38] N. F. Krasnov, *Aerodynamics.* Moscow: Mir Publishers, second ed., 1985.

[39] W. P. Walker, *Unsteady aerodynamics of deformable thin airfoils.* Ph.d. thesis, Virginia Polytechnic Institute and State University, 2009.

[40] L. Schwarz, "Berechnung der druckverteilung siner harmonisch sich verformungen tragfläche in ebener stromung," *Luftfahrtforschungr*, vol. 17, pp. 379–386, 1940.

[41] W. R. Sears, "Operational methods in the theory of airfoils in non-uniform motion," *Journal of the Franklin Institute*, vol. 230, no. 1, pp. 95–111, 1940.

[42] H. G. Küssner, "Zusammenfassender Bericht über den instationären Auftrieb von Flügeln," *Luftfahrtforschung*, vol. 13, no. 12, pp. 410–424, 1936.

[43] W. R. Sears and B. O. Sparks, "On the reaction of an elastic wing to vertical gusts," *Journal of the Aeronautical Sciences*, vol. 9, no. 2, pp. 64–67, 1941.

[44] C. R. dos Santos, F. D. Marques, and H. E. Taha, "Unsteady lifting line theory and dynamic stall modeling in finite wings," *Journal of Fluids and Structures*, submitted.

[45] J. D. Anderson Jr., *Introduction to Flight*. McGraw-Hill, 3 ed., 1989.

[46] B. Thwaites, "Approximate calculation of the laminar boundary layer," *The Aeronautical Quarterly*, vol. 1, no. 3, pp. 245–280, 1949.

[47] M. Goman and A. Khrabrov, "State-space representation of aerodynamic characteristics of an aircraft at high angles of attack," *Journal of Aircraft*, vol. 31, no. 5, pp. 1109–1115, 1994.